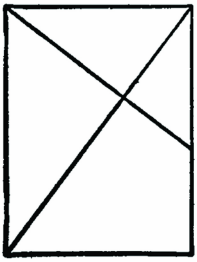
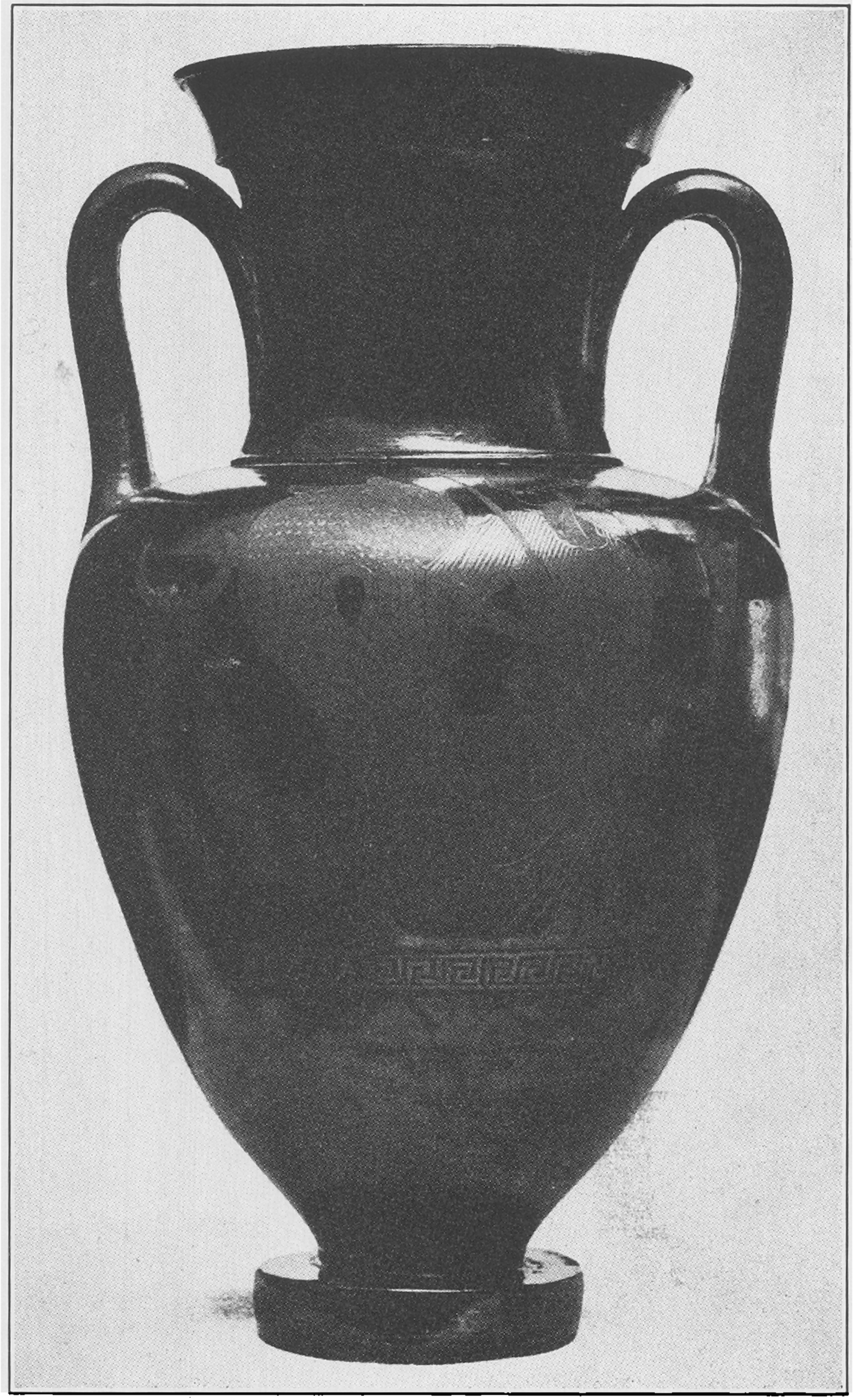
DYNAMIC SYMMETRY
THE GREEK VASE

BY JAY HAMBIDGE



YALE UNIVERSITY PRESS
NEW HAVEN CONNECTICUT AND NEW YORK CITY
LONDON · HUMPHREY MILFORD · OXFORD UNIVERSITY PRESS



AN UNUSUALLY HANDSOME NOLAN AMPHORA, FOGG MUSEUM AT HARVARD

A theme in root-two

CONTENTS

CHAPTER ONE: THE BASIS OF DESIGN IN NATURE

FOR the purpose of the present work, it will be sufficient to deal only with the conclusions obtained by the study of the bases of design in nature. There are so many fascinating aspects of natural form, so many tempting by-paths, that it would be easy to wander far from the subject now under consideration. Moreover, the morphological field has received attention from many explorers more gifted and better equipped to examine and interpret the phenomena of shape from a scientific point of view than the writer, whose training has been, and disposition is, merely that of a practical artist.[4] His working hypothesis, responsible for the material here presented, was formulated upon the assumption that the same curve persists in vegetable and shell growth. This curve is known mathematically as the constant angle or logarithmic spiral. This curiously fascinating curve has received much attention.[5] As a curve form, its use for purposes of design is limited, but it possesses a property by which it may readily be transformed into a rectangular spiral. The spiral in nature is the result of a process of continued proportional growth. This will be clear if we consider a series of cells produced during a period of time, the first cell growing according to a definite ratio as new cells are added to the system. (See Figs. 1 and 2.) The shell is but a cone rolled up. Fig. 1 represents the cone of such an aggregate, while Fig. 2 shows the system coiled.

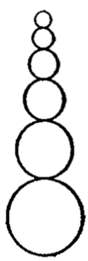


Fig. 1.

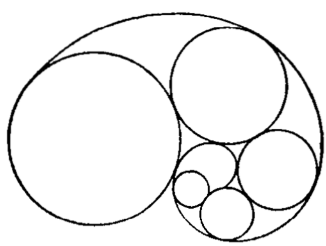


Fig. 2.

The curve of the coil is a logarithmic spiral in which the law of proportion is inherent. A distinctive feature of this curve is that when any three radii vectors are drawn, equi-angular distance apart, the middle one is a mean proportional between the other two; in other words, the three vectors, or the three lines drawn from the center or pole to the circumference, equi-angular distance apart, form three terms of a simple proportion; A is to B, as B is to C, and according to the "rule of three" the product of the extremes, A and C, is equal to the square of the mean. A multiplied by C equals B multiplied by itself. The early

Greeks covered the point geometrically when they established the fact that in a right triangle, a line drawn perpendicular to the hypotenuse to meet the intersection of the legs, is the side of a square equal in area to the rectangle formed by the two segments of the hypotenuse. (Fig. 3.)

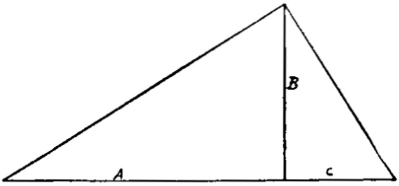


Fig. 3.

These three lines C, B, A, constitute three terms in a continued proportion.

When the three radii vectors are drawn from the center to the circumference of the shell curve, as in Fig. 4,

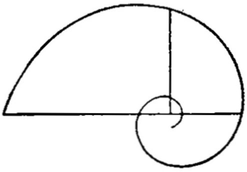


Fig. 4.

and these points of intersection with the spiral are connected by two straight lines, a right angle is created at C and a right triangle formed, ACB. (Fig. 5.)

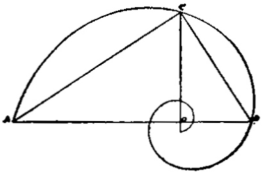


Fig. 5.

If the mean proportional line of this right triangle, ACB, that is, if the line CO be produced through the pole or center of the spiral to the opposite side of the curve, obviously another right angle is created as at B, and by drawing the line BD, the right triangle DBC is formed. (Fig. 6.)

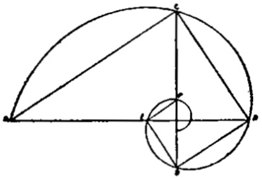


Fig. 6.

The process may be extended until the entire spiral curve has been transformed into a right angle spiral, as shown by the lines AC, CB, BD, DE, EF, etc., a form suggestive of the Greek fret. There now exists in the area bounded by the spiral curve a double series of lines in continued proportion, each line bearing the same relation to its predecessor as the one following bears to it.

As far as design is concerned, we may now dispense with the curve of the spiral. There have been extracted from it all essentials for the present purpose and there remains but the placing of the angular spiral within a rectangle. This may be done in any rectangle by drawing a diagonal to the rectangle and from one of the remaining corners a line to cut this diagonal at right angles. This line, drawn from one corner of the rectangle to cut the diagonal at right angles, is produced to the opposite side of the rectangle. (Fig. 7.)

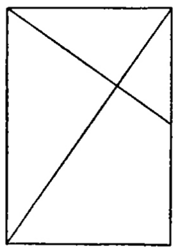


Fig. 7.

Such a line we shall refer to as a perpendicular, and in all cases it is drawn from a corner. It establishes proportion within a rectangle, and is the diagonal to the reciprocal of the rectangle. In Fig. 8, AB is a reciprocal rectangle and consequently is similar to the rectangle CD.[7]

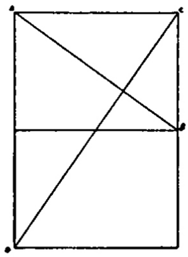


Fig. 8.

There exists a series of rectangles whose sides are divided into equal parts by the perpendicular to the diagonal. Take for example the rectangle in Fig. 9, where the line AB bisects the line CD, at B. In such a rectangle a relationship exists between the end and the side expressed numerically by 1, or unity, and 1.4142 (see Fig. 10) or the square root of two, and a square constructed on the end is exactly one-half, in area, of the square constructed on the side.

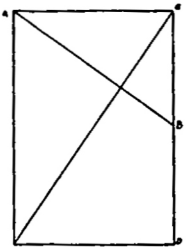


Fig. 9.

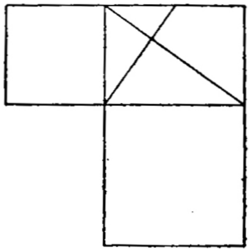


Fig. 10.

The student may draw all the rectangles of Dynamic Symmetry with a right angle and a decimally divided scale, preferably one divided into millimeters.

It will be noticed that the number 1.4142 is an indeterminate fraction. In other words, while the end and the side of this rectangle are incommensurable in line, they are commensurable in square.[6] This rectangle we may call a root-two rectangle. It is found to possess properties of great importance to design. It is the rectangle whose reciprocal is equal to half the whole.[7]

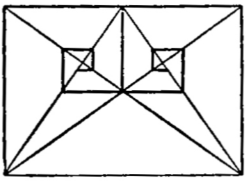


Fig. 11*a*.

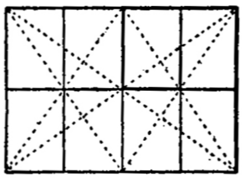


Fig. 11*b*.

Fig. 11*a* shows two perpendiculars in the rectangle, and rectangular spirals wrapping around two poles or eyes. If, as in Fig. 11*b*, four perpendiculars are drawn to the two diagonals, and then lines at right angles to the sides and ends through the intersections, the area of the rectangle will be divided into similar figures to the whole, the ratio of division being two.

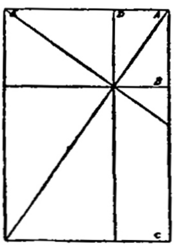


Fig. 12*a*.

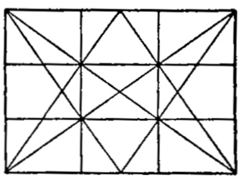


Fig. 12*b*.

If, instead of lines coinciding with the spiral wrapping, as in Fig. 11*a*, lines are drawn through the eyes, and at right angles to the sides and ends, the rectangle will be divided into similar shapes to the whole, with a ratio of three. (See Fig. 12.) AB is one third of AC, while AD is one third of AE.

A rectangle whose side is divided into three equal parts by horizontal lines drawn through the points of intersection of the perpendiculars and the sides of the rectangle has a ratio between its end and its side of 1, or unity, to 1.732 or the square root of 3. This is a root-three rectangle and has characteristics similar to those of a root-two rectangle, except that it divides itself into similar shapes to the whole with a ratio of 3. AB, BC and CD are equal. (Fig. 13.) Lines drawn through the eyes of the spiral divide this rectangle into four equal parts. The square on the end of this rectangle is one-third the area of the square on the side.

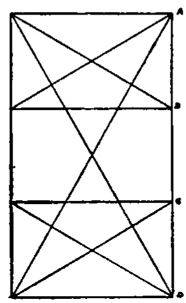


Fig. 13

A rectangle whose side is divided into four equal parts by a perpendicular has a ratio between its end and its side of one to two, or unity to the square root of four. This rectangle has properties similar to those of a root-two or a root-three rectangle, except that it divides itself into similar rectangles by a ratio of four, and the area of the square on the end is one-fourth the area of

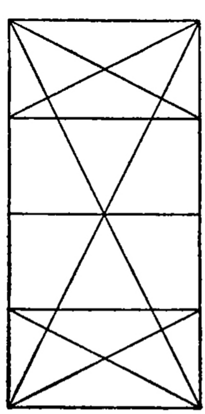


Fig. 14*a*.

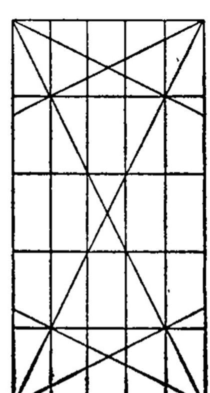


Fig. 14*b*.

the square on the side. This is a root-four rectangle. Lines drawn through the eyes of the spirals of a root-four rectangle divide the area into five equal parts similar to the whole. (Fig. 14*b*.)

A rectangle whose side is divided into five equal parts by a perpendicular has a ratio between its end and its side of one to 2.236, or the square root of five. This area is a root-five rectangle and it possesses properties similar to those of the other rectangles described, except that it divides itself into rectangles similar to the whole with ratios of five and six. A square on the end is to a square on the side as one is to five, that is, the smaller square is exactly one-fifth the area of the larger square. There is an infinite succession of such rectangles, but the Greeks seldom employed a root rectangle higher than the square-root of five.

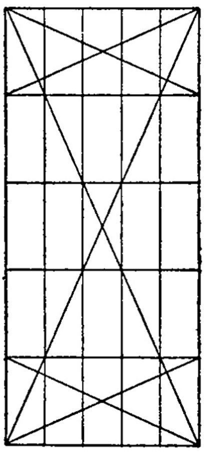


Fig. 15*a*.

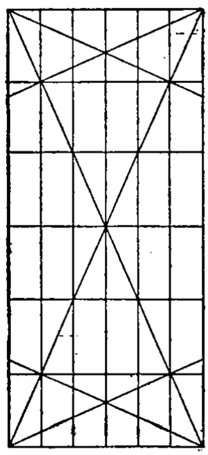


Fig. 15*b*.

The root-five rectangle, moreover, possesses a curious and interesting property which intimately connects it with another rectangle, perhaps the most extraordinary of all. To understand this strange rectangle, we must consider the phenomena of leaf distribution. This root-five rectangle may be regarded as the base of dynamic symmetry.[8]

Closely linked with the scheme which nature appears to use in its construction of form in the plant world is a curious system of numbers known as a summation series. It is so called because the succeeding terms of the system are obtained by the sum of two preceding terms, beginning with the lowest whole number; thus, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc. This converging series of numbers is also known as a Fibonacci series, because it was first noted by Leonardo da Pisa, called Fibonacci. Leonardo was distinguished as an arithmetician and also as the man who introduced in Europe the Arabic system of notation. Gerard, a Flemish mathematician of the 17th century, also drew attention to this strange system of numbers because of its connection with a celebrated problem of antiquity, namely, the eleventh proposition of the second book of Euclid. Its relation to the phenomena of plant growth is admirably brought out by Church,[5] who uses a sunflower head to explain the phenomena.

What is called normal phyllotaxis or leaf distribution in plants is represented or expressed by this summation series of numbers. The sunflower is generally accepted as the most convenient illustration of this law of leaf distribution. An average head of this flower possesses a phyllotaxis ratio of 34 x 55. These numbers are two terms of the converging summation series.

The present inquiry is concerned with only two aspects of the phyllotaxis phenomena: the character of the curve, and the summation series of numbers which represents the growth fact approximately.[9] The actual ratio can be expressed only by an indeterminate fraction. The plant, in the distribution of its form elements, produces a certain ratio, 1.618, which is obtained by dividing any one term of the summation series by its predecessor. This ratio of 1.618 is used with unity to form a rectangle which is divided by a diagonal and a perpendicular to the diagonal, as in the root rectangles. (Fig. 19.)

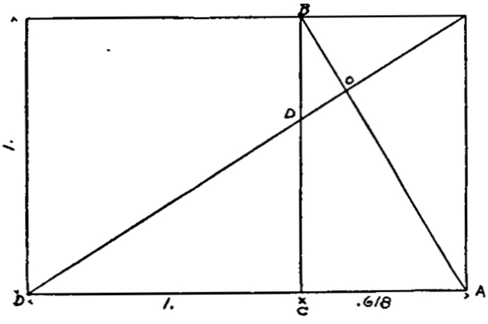


Fig. 19.

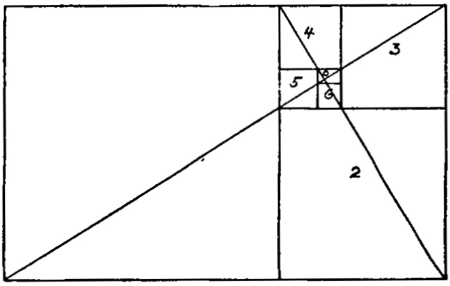


Fig. 20.

"A fairly large head, 5 to 6 inches in diameter in the fruiting condition, will show exactly 55 long curves crossing 89 shorter ones. A head slightly smaller, 3 to 5 inches across the disk, exactly 34 long and 55 short; very large 11 inch heads give 89 long and 144 short; the smallest tertiary heads reduce to 21 and 34 and ultimately 13 and 21 may be found; but these being developed late in the season are frequently distorted and do not set fruit well. A record head grown at Oxford in 1899 measured 22 inches in diameter, and, though it was not counted, there is every reason to believe that it belonged to a still higher series (144 and 233).

"Under normal conditions of growth the ratio of the curves is practically constant. Out of 140 plants counted by Weisse, 6 only were anomalous, the error thus being only 4 per cent." A. H. Church, "On the Relation of Phyllotaxis to Mechanical Law."[5]

Thus, we may call this "the rectangle of the whirling squares," because its continued reciprocals cut off squares. The line AB in Fig. 19 is a perpendicular cutting the diagonal at a right angle at the point O, and BD is the square so created. BC is the line which creates a similar figure to the whole. One or unity should be considered as meaning a square. The number 2 means two squares, 3, three squares, and so on. In Fig. 19 we have the defined square BD, which is unity. The fraction .618 represents a shape similar to the original, or is its reciprocal. Fig. 20 shows the reason for the name "rectangle of the whirling squares." 1, 2, 3, 4, 5, 6, etc., are the squares whirling around the pole O.

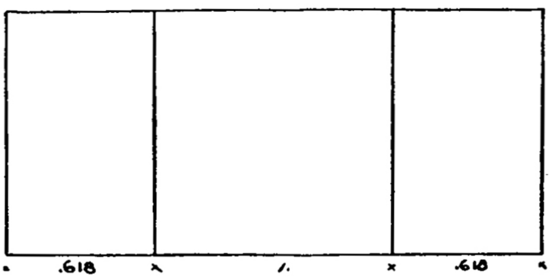


Fig. 21.

If the ratio 1.618 is subtracted from 2.236, the square root of 5, the remainder will be the decimal fraction .618. This shows that the area of a root-five rectangle is equal to the area of a whirling square rectangle plus its reciprocal, that is, it equals the area of a whirling square rectangle horizontal plus one perpendicular, as in Fig. 21.

The writer believes that the rectangles above described form the basis of Egyptian and Greek design. In the succeeding chapters will be explained the technique or method of employment of these rectangles and their application to specific examples of design analysis.

CHAPTER TWO: THE ROOT RECTANGLES

THE determination of the root rectangles seems to have been one of the earliest accomplishments of Greek geometers.[9] In fact, geometry did not become a science until developed by the Greeks from the Egyptian method of planning and surveying. The development of the two branches of the same idea went together. Greek artists, working upon this basis to elaborate and perfect a scheme of design, labored side by side with Greek philosophers, who examined the idea to the end that its basic principles might be understood and applied to the solution of problems of science. How well this work was done, Greek art and Greek geometry testify.

As early as the Sixth Century B. C. Greek geometers were able to "determine a square which would be any multiple of a square on a linear unit." It is evident that in order to construct such squares the root rectangle must be employed. We find the Greek point of view essentially different from ours, in considering areas of all kinds. We regard a rectangular area as a space inclosed by lines, and the ends and sides of the majority of root rectangles, because these lines are incommensurable, would now be called irrational. The Greeks, however, put them in the rational class, because these lines are commensurable in square.[6] This conception leads directly to another Greek viewpoint which resulted in the evolution of a method employed by them for the solution of geometric problems, to wit, "the application of areas."[10] Analysis of Greek design shows a similar idea was used in art when rectangular areas were exhausted by the application of other areas, for example, the exhaustion of a rectangle by the application of the squares on the end and the side, in order that the area receiving the application might be clearly understood and its proportional parts used as elements of design. If the square on the end of a root-two rectangle be applied to the area of the rectangle, it "falls short," is "elliptic," and the part left over is composed of a square and a root-two rectangle. (See Fig. 1*a*.) If the same square be applied to the other end, so as to overlap the first applied square, the area of the rectangle is divided into three squares and three root-two rectangles. (See Fig. 1*b*.) And, if the square on the side of a root-two rectangle be applied, it "exceeds," is "hyperbolic," and the excess is composed of two squares and one root-two rectangle.[11] (See Fig. 1*c*.)

This idea is quite unknown to modern art, but that it is of the utmost importance will be shown in this book by the analyses of the Greek vases.

Let us now consider various methods of construction of the root rectangles, and, of course, the whirling square rectangle. We will commence with the latter, which is intimately connected with extreme and mean ratio, a geometrical conception of great artistic and scientific interest to the early Greeks. Using dynamic symmetry, this problem of cutting a line in extreme and mean ratio may be solved through subtracting unity from the diagonal of a root-four rectangle: the Greek method was not essentially different. To the early geometers it was the cutting of a line so that the rectangle formed by the whole line and the lesser segment would equal the area of the square described on the greater segment.[5]

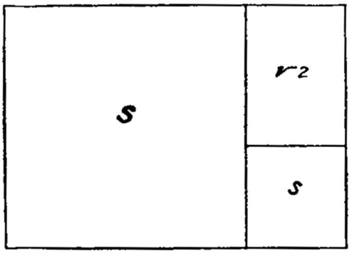


Fig. 1*a*.

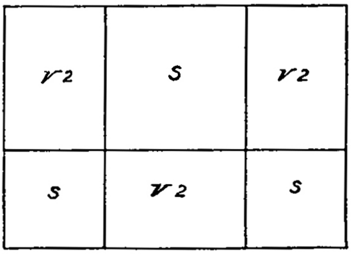


Fig. 1*b*.

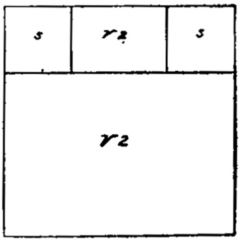


Fig. 1*c*.

Euclidean construction furnishes an easy method for describing not only the whirling square, but also the root-five rectangle, after the following manner: A square is drawn and one side bisected at A. The line AB is used as a radius and the semi-circle CBFD described. DE is a root-five rectangle. BC and DF are rectangles of the whirling square, as are also CF and BD. (Fig. 2.)

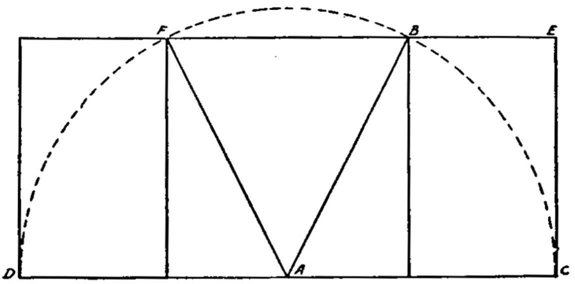


Fig. 2.

The relation of the rectangles, which have been described, to certain compound shapes derived from them will now be shown. If, in a rectangle of the whirling squares mapped out as in Fig. 3, a line parallel to the sides be drawn through the eyes A and B, it cuts from the major shape a root-five rectangle, *i. e.*, a square and two whirling square rectangles, C, D, and E,—D being the square. Fig. 4 shows how a line drawn through the eyes F and G, parallel to the end, defines also a root-five rectangle, C being the square. Obviously this may be done at either end and side, resulting in the determination of four root-five rectangles overlapping each other within the major shape. In a whirling square rectangle (Fig. 5*a*), if lines be drawn through the eyes A, B, C, D parallel to the ends, and A and B connected by another line, an area will be

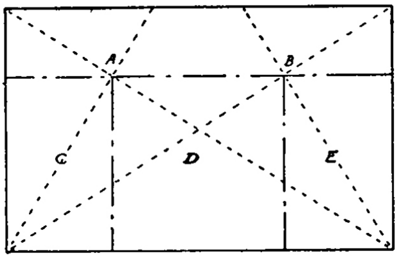


Fig. 3.

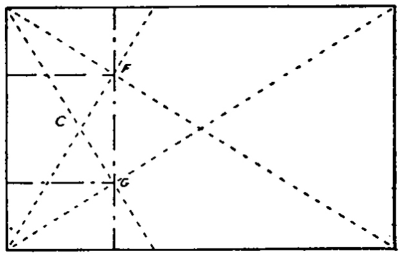


Fig. 4.

defined, composed of the square E and the rectangle F. This shape, composed of E and F, is numerically described as the rectangle 1.382. The square E is unity. The rectangle F is the fraction .382, this being the reciprocal of 2.618, *i. e.*, it is a whirling square rectangle, 1.618 plus 1. (Fig. 5*b*.) If this 1.382 rectangle is divided by 2, the shapes G, H (Fig. 5*c*), result and each is composed of a square and a root-five rectangle. 1.382 divided by 2 equals .691, which, divided into unity, proves to be the reciprocal of 1.4472, and .4472 is the reciprocal of root-five and is itself a root-five rectangle. Many Greek vases were constructed according to the principles inherent in this 1.382 shape.

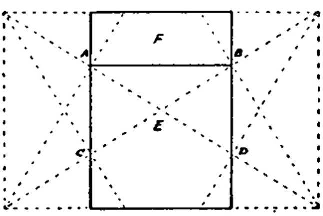


Fig. 5*a*.

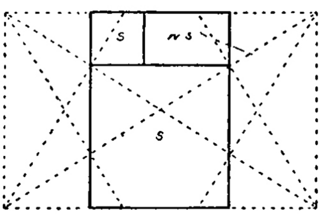


Fig. 5*b*.

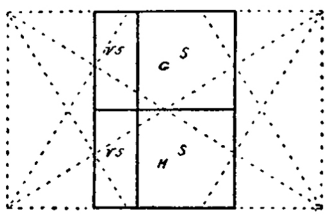


Fig. 5*c*

If a whirling square rectangle is subtracted from, or applied to, a square, the defect is .382 or a whirling square rectangle plus a square. (See Fig. 6.) .618 subtracted from 1. equals .382. If, as in Fig. 7, a whirling square rectangle is placed in the center of the shape 1.382, the "defect" area on either side is composed of a square and a whirling square rectangle.

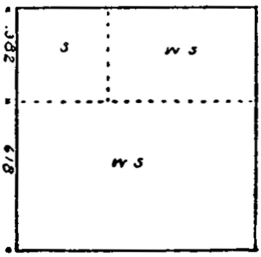


Fig. 6.

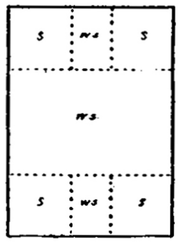


Fig. 7.

The reciprocal of 1.382 is .7236; .4472 multiplied by 2 equals .8944, and this result added to .7236 equals 1.618. (See Fig. 8.) The area of Fig. 8 is composed of two root-five rectangles, .4472 x 2, plus a .7236 shape.

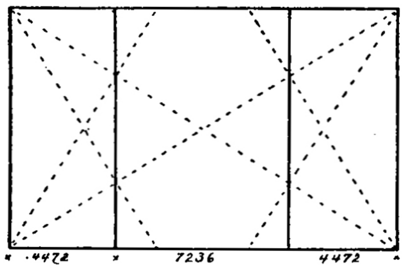


Fig. 8.

All of these shapes are found in abundance in both Egyptian and Greek art.

The square is considered the unit form or monad. "Iamblicus (fl. *circa* 300 A. D.) tells us that . . . 'an unit is the boundary between number and parts because from it, as from a seed and eternal root, ratios increase reciprocally on either side,' *i. e.*, on one side we have multiple ratios continually increasing, and on the other (if the unit be subdivided), submultiple ratios with denominators continually increasing." ("The Thirteen Books of Euclid's Elements," by T. F. Heath, Def. Book VII.)

The Reciprocal Ratios Within a Square

The root rectangles are constructed within a square by the simple geometrical method shown in Fig. 9. AB is a quadrant arc with center D and radius DB. DC is a diagonal to a square and it cuts the quadrant arc at F. A line, parallel to a side of the square, is drawn through F. This line determines a root-two rectangle and DE is its diagonal. A diagonal to a root-two rectangle cuts the quadrant arc at H. GD is a root-three rectangle, the diagonal of which cuts the quadrant arc at J. DI is a root-four rectangle and its diagonal cuts the quadrant arc at L. DK is a root-five rectangle and so on. All the root rectangles may be thus obtained within a square.

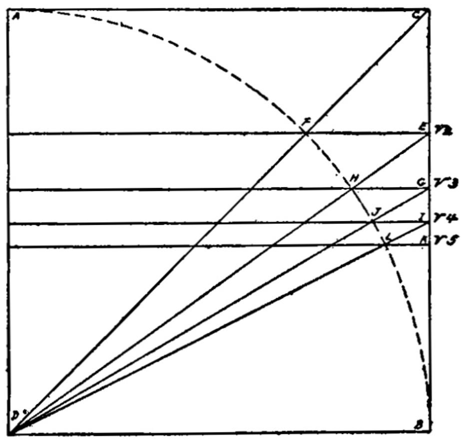


Fig. 9.

The root ratios outside of a square are obtained from diagonals, Fig. 10. AB, the diagonal of the unit form or square, determines the point C, the side of a root-two rectangle. The diagonal of a root-two rectangle, as AD, becomes the side of a root-three rectangle, as AE. AF, the diagonal of a root-three rectangle, becomes the side of a root-four rectangle, as AG. AH, the diagonal of a root-four rectangle, becomes the side of a root-five rectangle, as AI. AJ, the diagonal of a root-five rectangle becomes the side of a root-six rectangle, and so on to infinity. In any of these rectangles a square on the end is some even multiple of a square on the side. The square constructed on the line AC is double the square on AK; the square on AE is three times the area of the square on AK; the square on AG is four times the square on AK; the square on AI is five times the square on AK, etc. This was the Greek method of describing squares which would be any multiple of a square on a given linear unit.[5] The given linear unit is the line AK. The rectangles inside the square are the reciprocals of the rectangles outside the square. A root-two rectangle inside the square, for example, is one-half the area of the root-two rectangle outside the same square; a root-three inside, one-third of a root-three outside; a root-four inside, one-fourth of a root-four outside and a root-five inside, one-fifth of a root-five outside. And a reciprocal to any rectangle is obtained by drawing a perpendicular from one corner.

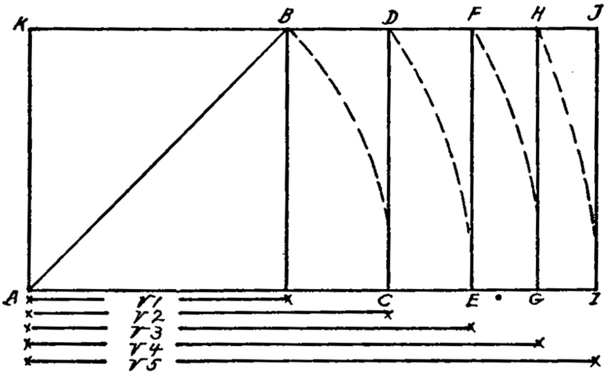


Fig. 10.

The whirling square rectangle and the root-five rectangle are placed within a square thus:

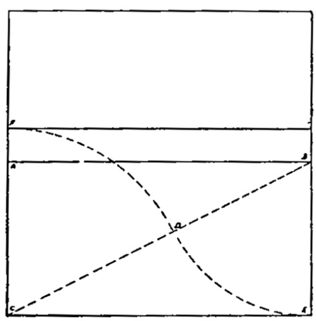


Fig. 11*a*.

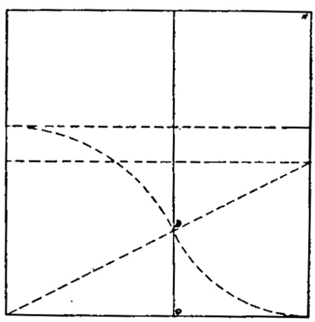


Fig. 11*b*.

The square is first bisected by the line AB, to obtain a root-four rectangle or two squares. From the diagonal of this rectangle CB, unity, or BE, is subtracted to determine the point D, and CD, furnishes the side of the whirling square rectangle FE. See Fig. 11*a*. A line drawn through the point D, parallel to a side of the square, determines the root-five rectangle GH. Fig. 11*b*.

In a whirling square rectangle inscribed in a square, if lines be drawn through the eyes and produced to the opposite side of the square, a root-five rectangle is constructed in the center of the square, see Fig. 12*a*. The area AB is this area, and if these lines be made to terminate at their intersection with the diagonals of the square, the whirling square rectangle CD, is defined as in Figs. 12*b* and 12*c*. That this construction was used by the Egyptians in design is shown by the bas-relief in the form of a square herewith reproduced:

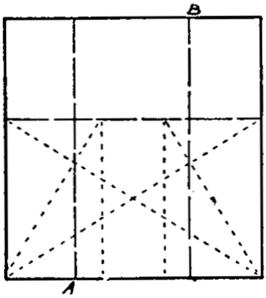


Fig. 12*a*.

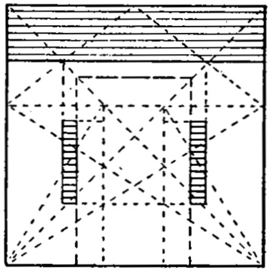


Fig. 12*b*.

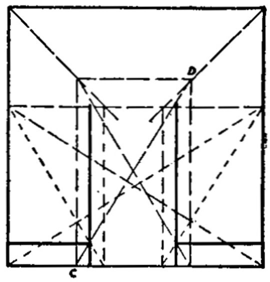


Fig. 12*c*.

When, as in Fig. 13, a whirling square rectangle is comprehended within a square, CD, the small square, AB, has a common center with the large square, CK, and if the sides of this small square, AB, are produced to the sides of the large square, CK, four whirling square rectangles, overlapping each other to the extent of the small square, AB, are comprehended in the major square. They are HK, EF, CD, and CJ, and the major square becomes a nest of squares and whirling square rectangles.

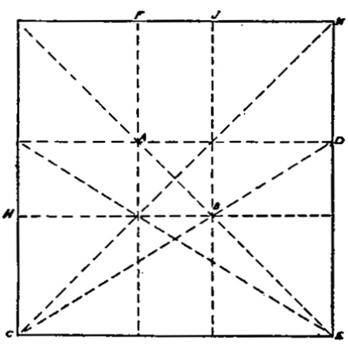


Fig. 13.

Analysis of the Egyptian bas-relief composition (Fig. 14) shows that its designer not only proportioned the picture but also the groups of hieroglyphs by the application of whirling square rectangles to a square. The outlines of the major square are carefully incised in the stone by four bars, two of which have slight pointed projections on either end. The general construction was that of *a* in Fig. 12. Spacing for additional elements of the design is shown in *c*, Fig. 12, while *b*, Fig. 12, exhibits the grouping of the hieroglyphic writing.

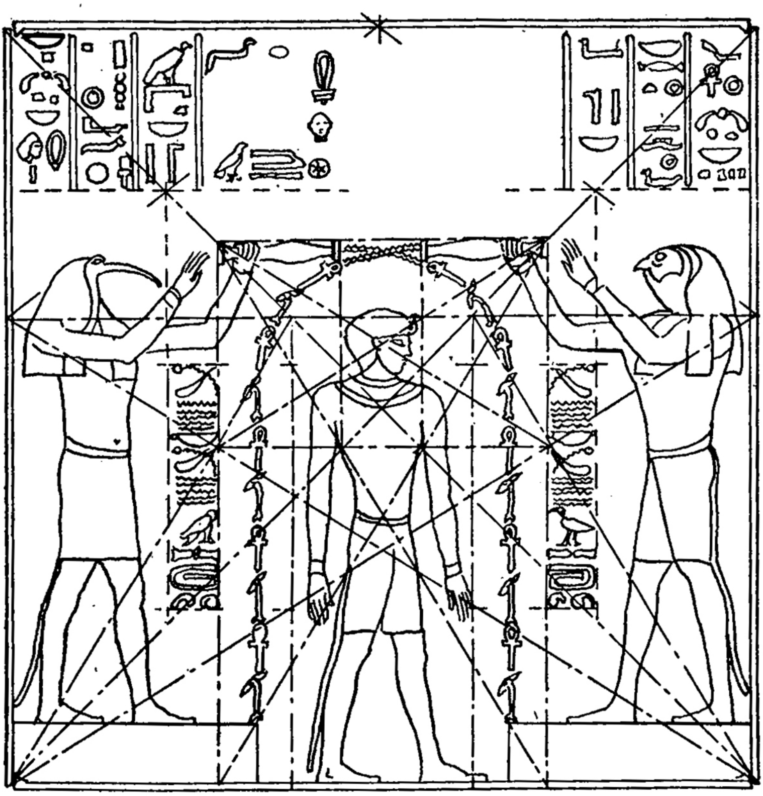


Fig. 14.

Another bas-relief from Egypt shows also how a square which is defined by bars cut in the stone at the top and bottom of the composition has its area dynamically divided for a pictorial composition. In this example the designer has used a root-five rectangle in the center of a square, Fig. 12*a*. The plan of this arrangement is obvious, Fig. 15.

A simple theme in root-two is exhibited in Fig. 16. A goddess is pictured supporting a formalized sky in the shape of a bar. The spaces between the bars on either side of the figure were filled with hieroglyphic writing. These have been omitted in this reproduction. The overall shape of this composition is a

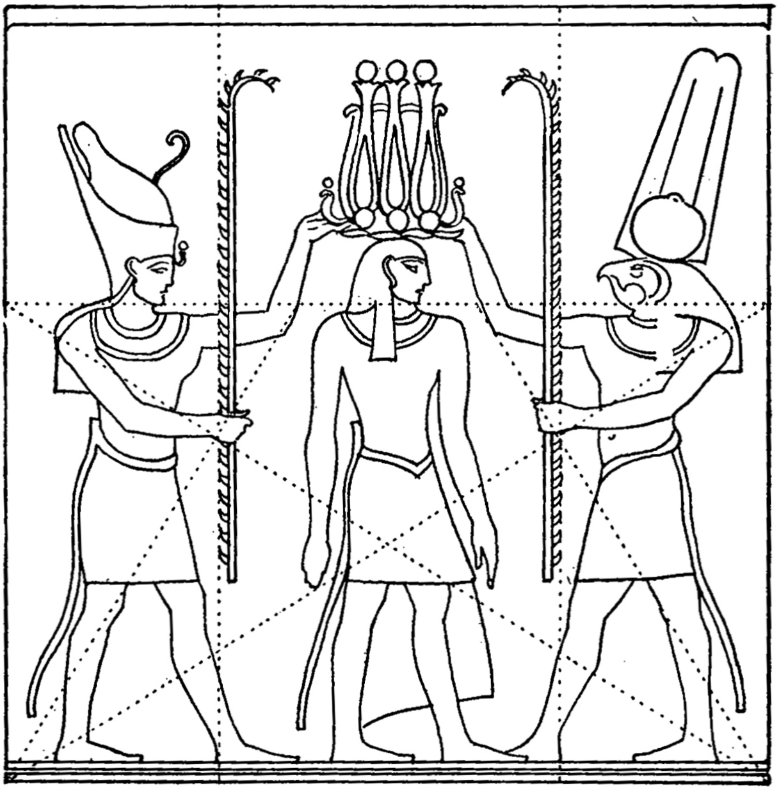


Fig. 15.

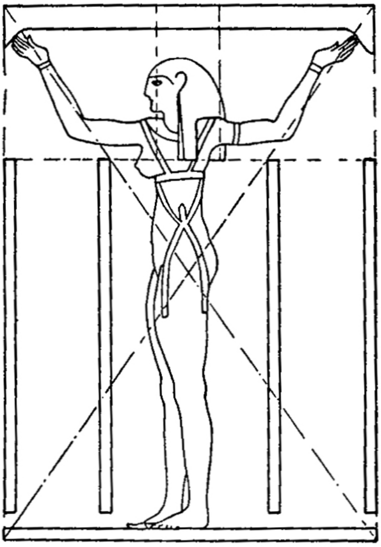


Fig. 16.

root-two rectangle and the simple method of construction is shown in Fig. 17. BC is a square and the side of the rectangle is equal in length to the diagonal of this square:

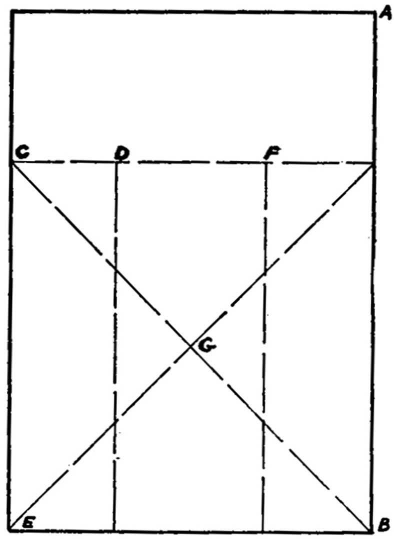


Fig. 17.

AB equals BC. DB and EF are root-two rectangles, the side of each being equal to half the diagonal of the major square, or the line BG. Diagonals to the whole intersect the side of the major square at the points D F.

Another theme in root-two is disclosed in Fig. 18. The general shape is a square, carefully defined by incised lines, as in the other examples.

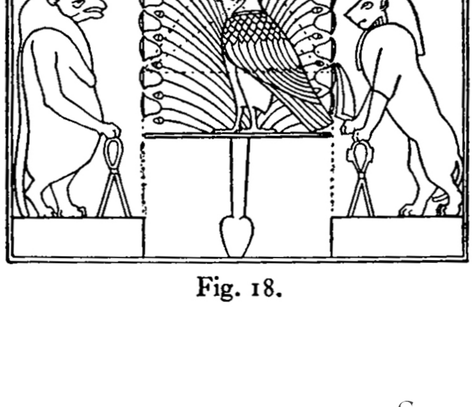


Fig. 18.

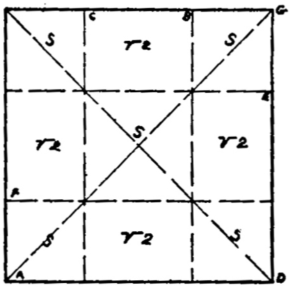


Fig. 19*a*.

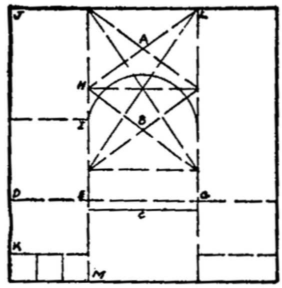


Fig. 19*b*.

The plan scheme of this design is shown in Fig. 19*a*. AB, CD, AE and FG, are four root-two rectangles overlapping each other in the major square, and the side of each, as CG, is equal to half the diagonal of the major shape. These rectangles subdivide the area of the major square into five squares and four root-two rectangles. In Fig. 19*b*, the use of this spacing, in its direct application to the design, is shown. The central portion of the major square, composed of the square HG and the root-two rectangle HL, is divided by the diagonals and perpendiculars of this rectangle. B is the center of the semicircle and BC is made equal to BA. This fixes the proportion of space to be occupied by the hawk and the field of formalized lotus flowers. MJ is composed of the two squares MD, DI and the root-two rectangle IJ. The square MD is divided into three parts and one of these parts forms the platform on which stands the hippopotamus god. This god is placed within the space KI. The same construction applies to the other side of the composition.

The examples of Egyptian bas-relief compositions described are, with one exception, arrangements within a square. These are used because of their obvious character. Like Greek temples and vase designs, the best Egyptian bas-relief plans are composed within the figures of dynamic symmetry, both simple and compound.

The Egyptians were regarded by the Greeks as masters of figure dissection. The rational combinations of form, which we may recover from their designs, confirms this and sheds some light on the significance of the ceremonial when "the king, with the golden hammer," drove the pins at the points established by the harpedonaptae, the surveyors or "rope-stretchers," who "corded the temple" and related the four corners of the building with the four corners of the universe.[2]