GOTHIC CATHEDRALS
AND
SACRED GEOMETRY

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VOLUME ONE

ALEC TIRANTI

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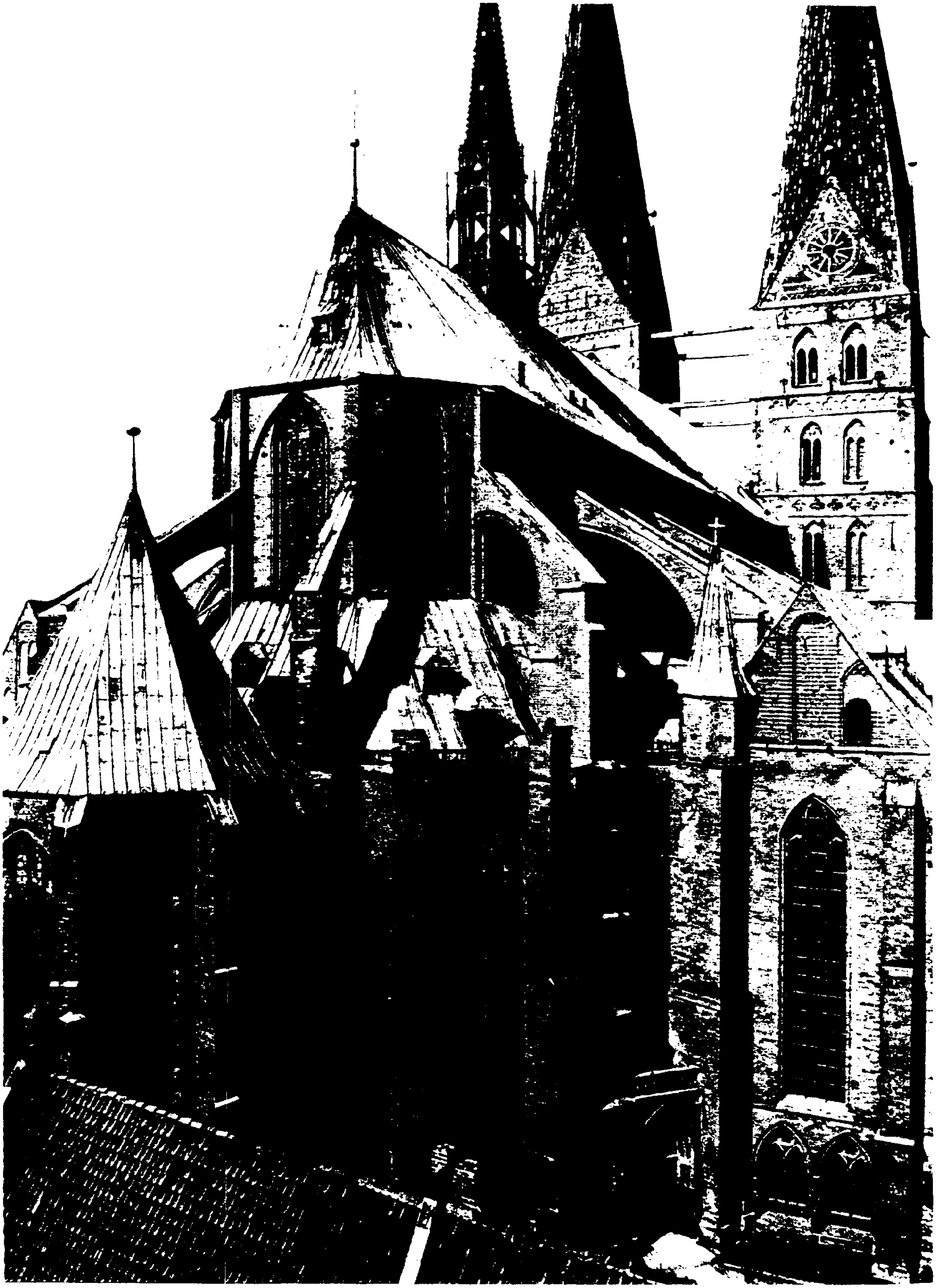

Frontispiece. St. Mary, Lübeck, *from the north-east*

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with reference to sources. Numbers in brackets relate to the Bibliography.

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[a] *Courtesy of Mr. A. Renger-Patzsch, Wamel in Westfalia.*

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[d] *The drawings are based on a survey, dated 1873, in the possession of the Hon. Society of the Middle Temple, London.*

[e] *After a survey dated 1930, supplied by the former Preussische Staats-Hochbauamt Stralsund.*

[f] *After a survey in the possession of the former Landesamt für Denkmalpflege, Schwerin.*

CHAPTER 1

Introduction

Geometry, as a lore of sacred numbers revealed in the shapes of planes and bodies, stands at the cradle of religious architecture, and is a holy art in itself.

Numerals are as old as language, and *tale* and *telling* stand for *speaking* and *counting* at one and the same time. But the age when numbers were invested with a mystic significance probably did not come before the rise of priesthoods for the worship of the gods.

In primitive religious communities, it is the first office of the priests to invoke blessings or curses, to solicit the favour or allay the wrath of the godhead. Religion at an early stage is mainly a mastery of rites, such as sacrifices, consecrations, conjurations and the like, and in all these the knowledge and control of the right numbers, propitious or ominous, is vital. Sacrifices have to be made, processions, dances, funerals have to be arranged, all observing the correct numbers and sequences so that the desired assistance of the gods is secured.

The beginnings of astronomy and chronology enter the scene when the right days and hours have to be selected for sacramental acts, and when the favourable or unfavourable conditions of a place have to be discerned by observation of the celestial bodies. Thus the priests, guardians of all learning, become the earliest astronomers. Evidently, they also act as geometers when places of worship are consecrated and adorned, and an area has to be surveyed under certain rites and enclosed to a definite shape with a ritual aspect in relation to the cardinal points.

Naturally, just as astronomy at its origin, and for long ages to come, is indistinguishable from astrology, so the beginnings of geometry ('land-surveying') cannot be separated from magic. It is all part of the holy 'technique' of the priests and therefore partakes of the veneration due to every ritual.

Without attempting to visualize more clearly such primitive, yet long-lived conditions, we will now try to find a thread leading from primeval examples of religious architecture to the creations of high civilizations, always remembering the role played by geometry in the procedure of priestly architects.

In the neolithic period, holy places are fenced in by circles of roughly worked pillars — the Cromlechs. For example, at Stonehenge there are three concentric enclosures around an altar. A processional path, leading to the centre, is orientated toward the point where the sun rises at the summer solstice.

The Menhirs are single pillars, raw blocks erected as monuments of an unknown meaning. They are the ancestors of a large and ubiquitous progeny. Whatever the reasons, it is a fact that man, primitive and civilized, to this day exults in raising memorial columns. As soon as stone-masonry is understood, we find the application to monumental columns of the simplest geometrical forms, such as the circle and the square, and of a primitive stereometry. The obelisk of Shelmaneser III in the British Museum is a square prism with a stepped finial. The Egyptian obelisks are slender truncated square pyramids, each crowned by a miniature pyramid. The celebrated Irish towers are slightly tapering cylinders, topped by regular cones.

Another class of stereometric monuments may be described as artificial mounds, such as the temple terraces of the Mesopotamians and Americans. Most emphatically stereometrical are the pyramids of the Nile valley, which are tombs of god-kings, and the Buddhist stoopas, which are gigantic reliquaries. The stoopa of Sanchi for example is essentially a hemisphere resting on a cylindric base. The Kaaba at Mecca is a primitive temple in the shape of a regular cube, surrounded by a circle of pillars connected by beams (a 'Cromlech'). All these instances, except the last, are compact bodies with no interior besides sepulchral cells, corridors, or stairs.

When images, arks, or other sacred implements have to be housed, we find the beginnings of *spatial* structures and of temple architecture in its usual sense. Here, it seems, the primordial creative impulse is not of a strictly geometrical description. Naturally, the earthly abode of the godhead is fashioned after the human habitation, the house, cabin, or tent of the nobles, be this of a circular, oblong, or other type. Thus the Greek and Chinese temples clearly preserve the memory of kings' halls. The type of the Semitic temple is also derived from a palace, incorporating the innermost hidden chamber of the ruler, an ante-room or audience-hall, inner courts, forecourts, gate-houses, etc. But here it should be remembered that the temple of Solomon was an arrangement of three squares of twenty cubits each, one for the Holy of Holies, two for the Outer Sanctuary. The former seems to have been a regular cube twenty cubits high, the latter had a height of thirty cubits, i.e. one and a half times its width. Here then, one might say, geometry was introduced as an afterthought. This, we believe, can be formulated as a law: whether the origin of a type of sacred architecture can be traced back to a type of human habitation, of an assembly or tribunal hall (the 'basilica'), or possibly to an abstract formal ideal that is not connected with any profane use — in any case, only the rule and efficacy of sacred geometry will make that fabric a holy shrine inhabited by the god and a place worthy of sacred acts. Geometry, in a specific and circumscribed sense, has to govern the design, and in turn such geometry is holy by virtue of its power to please and attract the godhead.

If we found the origin of sacred geometry in primitive sacerdotal rites, in magic and conjuration, we should now add that on a higher level, in the fully developed religions of advanced civilizations, that conception is superseded by the sublime idea that the sacred building is destined to represent the imagined structure of the *Universe* as the domain of the godhead. Thus, to choose the most obvious instance, the domed temple in its geometric form clearly symbolizes the terrestrial circle and the celestial sphere above. The precinct of the Etruscan temple was the imagined counterpart to a circumscribed area of the starry sky. Any sort of orientation is a deliberate endeavour to fit the holy place into a preconceived world-order. The Zigurrat of Sargon's palace at Khorsabad was ascended by a square spiral ramp rising in seven steps. Each step carried the colour of one of the seven planets : thus that temple tower reflected the cosmic order of the stars. In *The Art and Architecture of India* by Benjamin Rowland (London 1953) we read with regard to the Buddhist temples of Cambodia (p.229) : 'The temple-mountain . . . is the importation of the old Indian concept of *pratibimba,* the making of either sacred mountains or unseen celestial regions in architectural constructions'; and concerning the Barabudur of Java (p.260) : 'This is an exact and accurate architectural representation of the sky as a solid vault covering the world mountain.' The tomb temples of the Fifth Dynasty at Abusir have a gate-house on the bank of the Nile from which a long corridor over a causeway leads up to the peristyle-court of the temple proper. Behind the temple rises the pyramid and so terminates the axis. In this grandiose disposition we seem to see an image of the way the soul transmigrates from the Hither Shore along the paths and the stream of the Nether World, toward the re-union of the deified king with his father, the Sun, whose symbol is the golden 'pyramidion' that covers the tip of the pyramid : the Other World is mirrored in this architectural creation.

The Universe, created by the god as a rational and therefore mathematical conception, the highest manifestation of divine wisdom, is imitated, in a mystic way, by the sacred building and its precinct which thus become a reflection of divine order, harmony, and beauty. Certain numbers and geometric figures or bodies are now sacred on the ground of the symbolic meanings attached to them. Numbers such as Three or Seven are hallowed as references to cosmologic, theogonic, or astrologic concepts, and geometric figures, for instance the regular polygons, because they exhibit symbolic numbers.

So much, we think, can be confidently said in general. In chapter 36 we will try to reconstruct another cosmic concept, that which is expressed by the specific abstract geometry we are going to scrutinize.

* * *

Our proper subjects are selected specimens of Gothic churches, most of which belong to the mid-Gothic style of the thirteenth and fourteenth centuries, and some to the Late Gothic style. They are 'basilican' churches (and a few 'hall-churches') of three aisles, mostly cruciform.

The direct descent of this type from the early Christian basilica of Roman or Ravennatic type is well known and can be traced in a continuous line. But the curious result of our investigations, which must be anticipated here, is that *in respect of the principles of 'sacred geometry'* the vaulted masterpieces of the High Middle Ages appear to have been derived not from the old unvaulted basilica, but from *vaulted* and *centralizing* fabrics, to wit the early baptisteries and sepulchral churches of circular and polygonal plans which were later, from about A.D. 500, followed by parish and cathedral churches on similar lines, notably in the Byzantine sphere of influence.

Perhaps it will be relevant to quote here W. R. Lethaby, *Medieval Art* (London 1904/11/12). After referring (p.81) to J. Strzygowski's classification of ancient Christian churches into (1) basilicas, (2) octagons, (3) domed basilicas, and (4) domed cross churches, Lethaby goes on (p.84ff.) to state that *in the west,* by means of repeated waves of eastern influence, the 'congregational, basilican, or ship type' of plan was brought together with the 'martyrium' of circular or cross type, usually entirely vaulted, and finally the aisled cross church of Romanesque type was reached as a synthesis. This may be allowed to support the above preliminary statement, until we can substantiate it through our diagrams.

Within the scope of this book it will not be possible even to sketch a comprehensive family-tree covering, from our point of view, the whole period of church design from the fourth century up to the birth of the Gothic style. Indeed, a similar enterprise would require the work of a lifetime. Nevertheless, before we reach our specific era, the Gothic age, we shall, as a foundation, study and pursue toward their origins the geometric principles of vaulted centralizing fabrics. It will be found that this school of thought even goes back to pre-Christian times. Analyses of such comparatively plain systems will prove useful for the understanding of those longitudinal churches which form the end of our search.

CHAPTER 2

Something about 'Proportions'

Most of the monuments cited in the first chapter are solid bodies whose geometric form reveals itself at the first glance. Their 'abstract' geometry is one with their visible, sensuous architecture. Such geometric lucidity may also be apparent in *spatial* structures, such as the Holy of Holies in the temple of Jerusalem — a pure cube. Another instance is the interior of the Pantheon in Rome (cf. chapter 6 following) which, as is well known, consists of the dome (a pure hemisphere), and a drum or cylinder whose height equals its radius, so that the dome, doubled into a full sphere, would touch the floor of the temple.

Evidently, when richer shapes of rooms, and consequently of the external architecture, are evolved, such a clarity of the geometric disposition cannot, as a rule, be maintained. When, for instance, niches or ambulatories are added to a circular or polygonal room, the basic geometric schemes for plan and elevation have to be enriched and articulated. Thus the transparency of the basic scheme may be dimmed, and in buildings such as the Minerva Medica in Rome or San Vitale of Ravenna, an analysis will fail to result in plain ratios. For example, the width of the central room may not stand in simple and rational relations to the overall width including the niches or aisles, or to the heights of the several parts, and so on. At least, this is the general experience with historic buildings within our scope. It is difficult, therefore, to find clear relations and establish consistent rules, and the present knowledge of the whole subject is unsatisfactory and in need of further elucidation.

What are the current opinions about geometric formative principles in historic architecture? In contemplating old monuments, the eye seeks first the numerical ratio between length, breadth, and height. Such ratios are commonly called 'proportions,' and in the usage of the language this word embraces all the geometric relations that are or may be inherent in an architectural design. Against this, we must here proclaim the thesis that 'proportions,' so defined, are not necessarily the *whole* of geometric relations known to and used by ancient architects. While such premeditated and inter-related proportions are certainly always present where an ancient work deserves the name of architecture, we shall realize that they alone do not, for our church buildings, constitute a comprehensive metrical system : that which we chose to call, for want of another familiar term, 'sacred geometry.'

A short retrospective view takes us to the Italian Renaissance. The illustrations of this chapter, which we borrow from August Thiersch (*Handbuch der Architektur*),[1] are almost self-explanatory. Generally speaking, the harmony of a design is here found in the rational repetition and modification of *rectangles* whose long and short sides are of the same relation or relations, in plan, section, and elevation.

Thus in the façade of the Strozzi
2 Palace in Florence, the standard window, completed into a rectangle, including the arch, has the proportion 7 :13. Forming a larger rectangle out of such a window, the two adjoining piers (each equal to the window in width) and the wall above up to the frieze of the main cornice (equal to the window in height), we get a proportion of 21 (breadth) : 26 (height). That proportion—21 (height) : 26 (breadth) — also governs the whole
1 front. The same figure shows how the heights of the ground floor and first storey stand to their string-courses in the same relation 7 :1 as the height of the whole front to the cornice including
3 the frieze. The rectangle of the plan is 7 :5, that of the courtyard 7 :4, and this latter ratio applies to the solid wall enclosing the court rectangle as well as to the inner face of the court arcades. Thus the whole building is a rectangular block governed by a few 'proportions' which are inter-related and repeated at varying sizes.

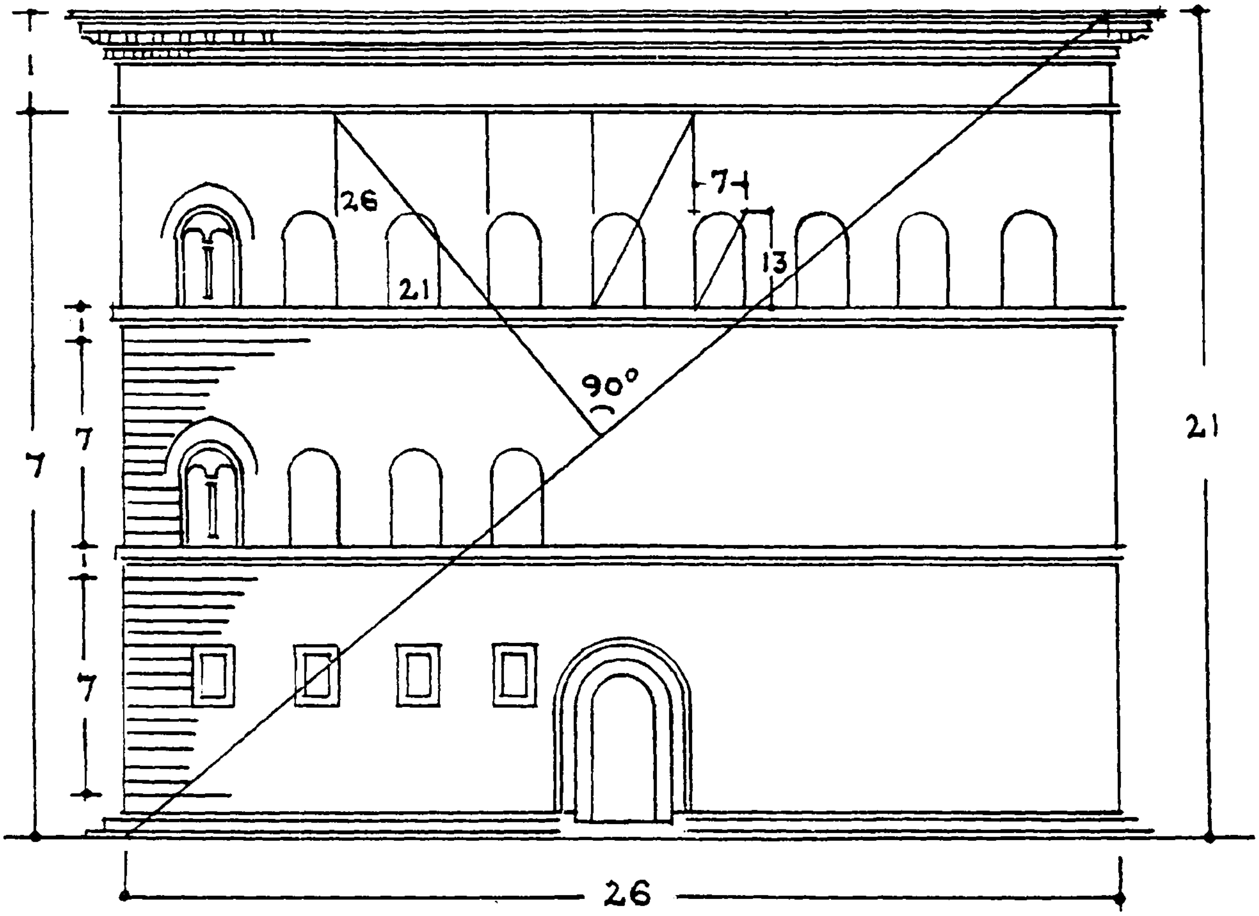
The interplay of such proportions, expressed by parallel diagonals (or diagonals perpendicular to each other), is also seen in the part elevation of the Farnesina, a section through the great 4
hall of the Massimi Palace, and the 5
elevation of St. Peter after Michel- 6
angelo's design (all in Rome). Thus in figure 4 we find rectangles 15 :7 and 2 : 1, in figure 5 rectangles 3 :2 and 2 :1, and in figure 6 rectangles 4 :1, and lineal relations 3 : 1.

At the Tempietto di San Pietro in 7
Montorio, Rome, parallel diagonals demonstrate that the elevation of the attic supporting the dome is a rectangle similar to the Tuscan order of the main storey, the proportion being 15 to 8.

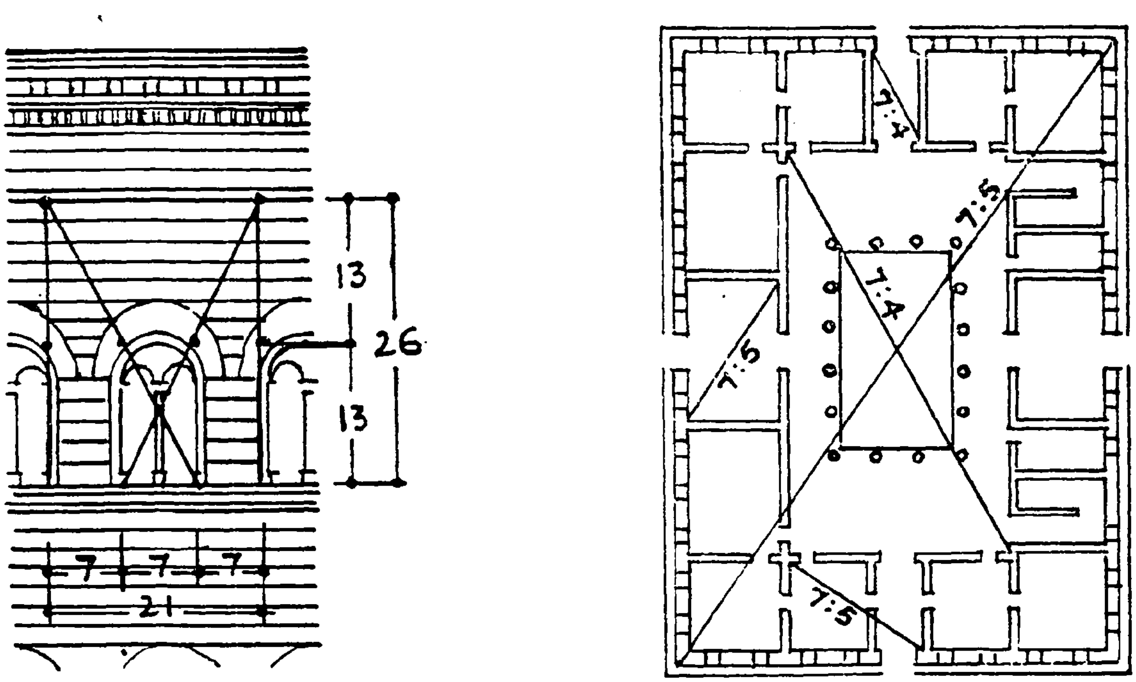
This then, the judicious repetition of similar rectangles, appears to constitute 'proportions' in the proper sense. They are certainly common to the Greco-Roman and Renaissance schools of architecture. Viollet-le-Duc applies the same principle to mediaeval churches (*Dictionnaire Raisonné*, volume 7, under 'Proportions').

Such relations and similarities are convincing, their aesthetic value is obvious. They are certainly sufficient to cover all dimensions and subdivisions, down to small details, of designs of classical purity and simplicity such as the Strozzi Palace, or even richer designs, as long as the right angle rules absolute. (In parenthesis, perhaps one might discover that all secular architecture since the Renaissance down to the beginning of the nineteenth century was essentially designed on that principle.)

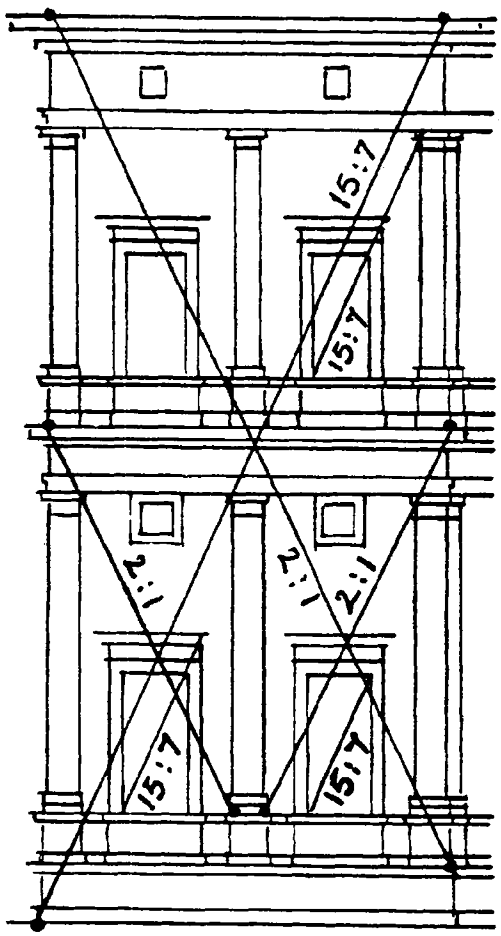
[1]Incorporated in J. Burckhardt, *Geschichte der Renaissance*; see Bibliography.



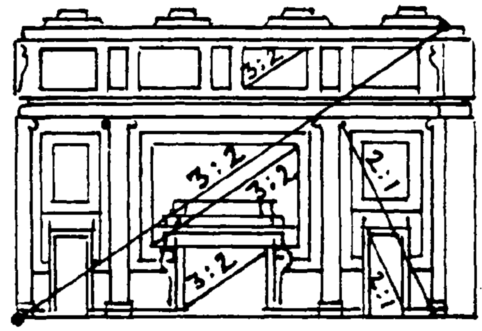
1. *Strozzi palace, Florence. Façade.*



2—3. *Strozzi palace, Florence. Detail and Plan.*



4. *Farnesina, Rome.*



5. *Massimi palace, Rome. Great hall.*

Renaissance churches may happen to follow such lines. A glance at the plan of Brunelleschi's San Lorenzo at Flor-
8 ence reveals a plain grid of squares which supplies all measures of the nave, the aisles, the choir, etc. The square bay of the aisles forms a modulus from which the whole plan is evolved by mere multiplication.

But evidently such methods of design will fail as soon as certain more complicated ideas have to be expressed, and most assuredly in all cases where a *centralizing* plan is adopted : that is when the rule of the circle and its derivatives is present.

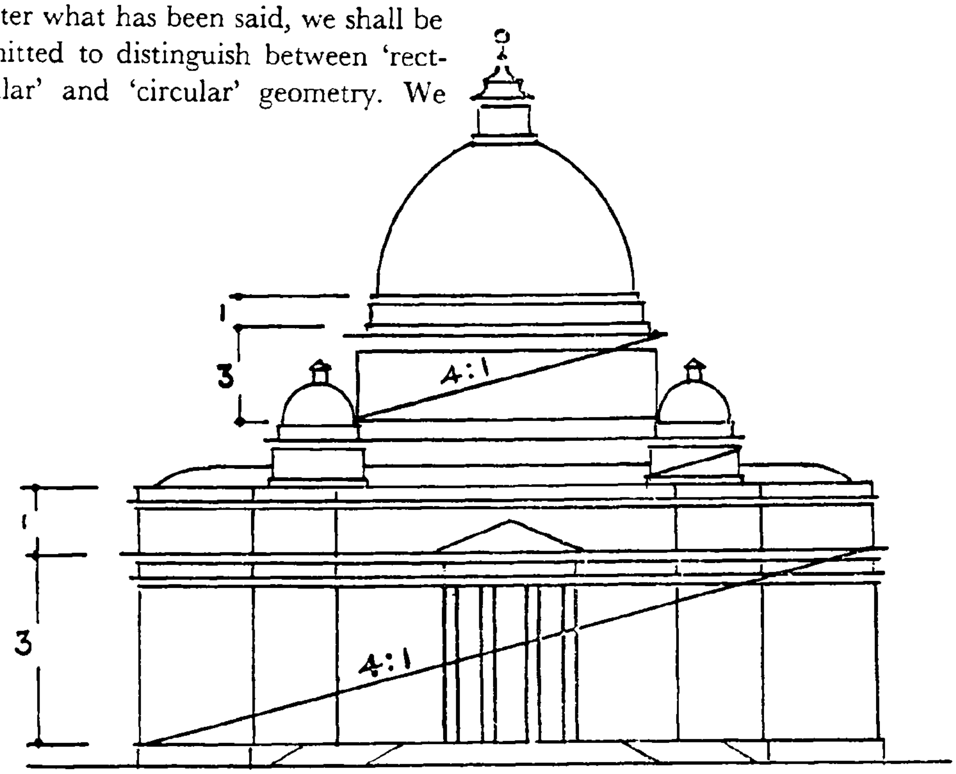
Figure 9 illustrates the plan of Bra- 9
mante's San Pietro in Montorio (cf. figure 7) as it would have been if fully executed to the architect's intentions. Here an attempt to apply 'similar rectangles' would be futile. The whole layout is clearly based on the circle and regular polygons. As the sixteen columns of both circular porticos suggest, the geometric analysis leads to an *octagon* as the basic figure. Since we shall deal with octagonal systems more fully, we now but briefly mention the concentric octagons — four in all — which are evolved from each other and supply the perimeters of all the circles that occur in the plan, and all important diagonals. Let us try to visualize the genesis of such a design. Bramante's original idea, his sensuous conception, was a round domed chapel with a surrounding colonnade after the model of Roman circular temples, with the addition of a ring-shaped court and a concentric outer portico. This idea, no doubt, was first laid down in a rough sketch. Then, as a geometric skeleton appropriate to that idea, the architect found or invented the described system, and by adjusting to

this his measures, he completed the design.

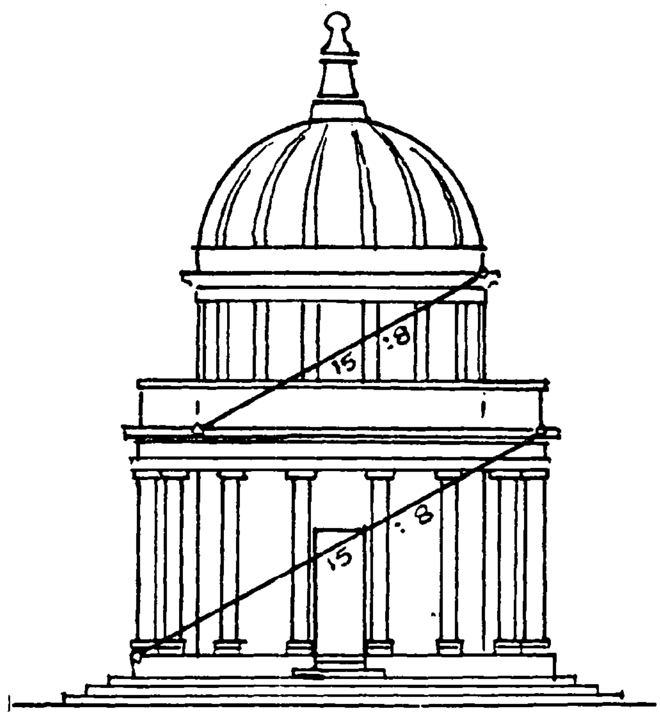
In our opinion, such a procedure does not seem to detract from Bramante's merit as a creative artist. We shall have to accept the fact that architects of bygone periods resorted to pure geometry in order to 'underpin' their visible and sensuous formations. Their line of thought was different from ours. Church building was worship. Aesthetic ideals, as we visualize them, were during the Middle Ages subordinated to a theology, and during the Renaissance to a religious philosophy, which claimed supreme importance. Expression of such superior principles was sought in the application of some abstract geometry, highly symbolical of course. More about this point will be discussed later.

After what has been said, we shall be permitted to distinguish between 'rectangular' and 'circular' geometry. We now venture to say that, during the Middle Ages, a church dedicated to Omnipotence had of necessity to be conceived and planned according to a 'hierarchical' geometry where all parts and measures radiate from a central point which marks the apex and focus of the design : in other words, a circular or, if we may so put it, a centripetal geometry. Compared with this ideal, a merely rectangular or 'proportional' geometry would appear secular and profane. Therefore we may expect that such rectangular relations are, in mediaeval church design, admitted in an ancillary role only.

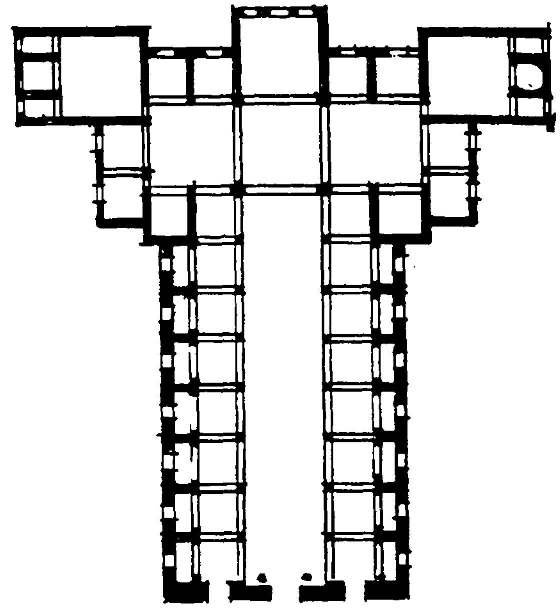
We can now leave our general discussion and confine our reasoning to the evolution leading up to the Gothic church.



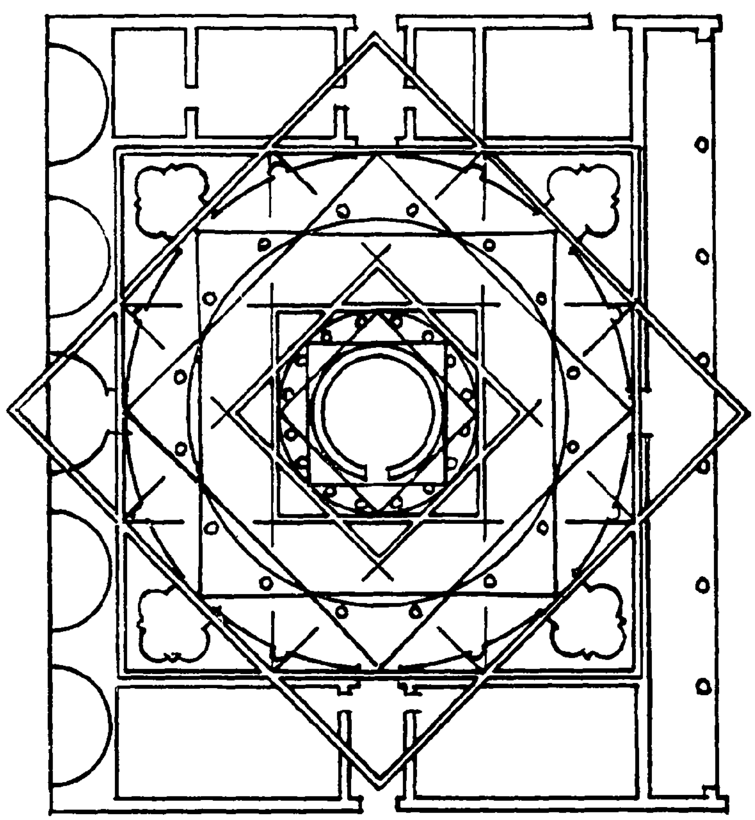
6. *S. Pietro, Rome.*



7. *S. Pietro in Montorio, Rome. Elevation.*



8. *S. Lorenzo, Florence.*



9. *S. Pietro in Montorio, Rome. Plan.*

CHAPTER 3

Polygons and Polygrams

Long before man has even the fundamental conceptions of geometry, Nature shows him the images of the circle and the polygon. Sun and moon are circles. The pupil of the human eye is a circle. A drop hitting the surface of a pond causes the water to ripple in concentric circles. The calyces of many flowers form circles or regular polygons. In the animal kingdom, starfish, jellyfish, sea-urchins, and other marine forms present themselves; and most impressively the cell of the honeycomb is a silent lesson in polygonal geometry. To this add minerals such as the basalt and many crystals, and it is understandable that, from the beginnings of pottery and textile art, circular and polygonal forms provide a theme for early artisans.

Polygrams are amplifications of regular *polygons* that are probably first discovered in plaiting and weaving. Geometrically, they are arrived at when the sides of a regular polygon (other than the triangle and the square) are produced until they intersect. Thus the regular pentagon is amplified into the pentagram, the hexagon into the hexagram, and so forth. Such figures can be formed of branches or threads. Then the polygrams of even numbers, the hexagram, octagram, decagram, dodecagram, etc., will be plaited of two threads, whereas those of odd numbers, the pentagram, heptagram, enneagram, etc., require but one thread which is intertwined so as to recur in itself to its beginning. Such textures occupy the fancy of primitive man and are used for magical operations, for 'tying knots,' for 'binding' and 'unbinding spells.'[2] Everyone knows the pentagram, or 'Solomon's seal,' and the hexagram, or 'David's star,' as magic symbols.

We will now, without giving too much theory, examine the geometric properties of polygonal and polygrammatic figures as far as they will interest us later.

The plain circle, although it suggests most impressive corporeal and spatial creations, is in itself inadequate for articulation. For instance, as soon as a circular room is to be enlarged by niches, it is found helpful to insert radii,

[2]'Anything twisted or knotted was of the nature of an amulet and exercized a deterrent effect on demons'—W. O. E. Oesterley, *Judaism and Christianity*, London 1937, volume I, p.208.

chords, or tangents, and therewith polygonal elements are introduced.

We by-pass the pentagon and its derivative, the decagon, because we shall encounter them very rarely in what we have to discuss. The same applies to the 'Golden Section' which is inherent in the decagon. Twice only,
XLVIII at the choir plans of Reims and Séez
25 Cathedrals, do we find a decagonal feature. We will anticipate here our opinion that decagonal geometry in these cases plays a secondary role and only governs the radial disposition of the chevet whose concentric peripheries are derived from the octagonal and dodecaïdal geometry of the frame figure.

When, in 1391, Gian Galeazzo Visconti, lord of Milan, summoned French and German architects to consult with their Milanese colleagues about the design of the cathedral, opinions turned about the question whether the church should be devised *ad quadratum* or *ad triangulum*. There can be no doubt that this antithesis is equivalent to the alternative between *octagon-octagram* (which, as will be shown presently, can be derived from the square) and *hexagon-hexagram* whose connection with the regular triangle is obvious. It is these figures that will command our attention for almost the whole of this treatise. Of the two systems, the former will prove to be of the highest importance; square, octagon and octagram, as polygons with two equivalent axes at right angles, are evidently the most adequate and inspiring figures for planning on a grand scale and in this respect superior to triangle, hexagon and hexagram. It may be mentioned that enneagons, dodecagons and heccaidecagons (polygons arrived at by dividing the circle into 9, 12, or 16 parts respectively), where they occur, are easily evolved from polygons with 3, 6, or 8 angles.

The regular *hexagon*, as young school- XVIIa
boys find out for themselves, is formed by dividing the periphery of the circle by its radius **r** which is contained six times in the periphery. By drawing the three centric diagonals, the hexagon divides into six regular or equilateral triangles whose side equals the radius **r.**

By producing the six sides of the
hexagon until they intersect, we get an XVIIc
extended figure, the *hexagram*, which is the hexagon plus six 'teeth,' i.e. six regular triangles attached to its sides, each of the teeth being equal to the six triangles inside the original hexagon. Another way of obtaining a hexagram
is to turn a regular triangle by 60° XVIIb
about its centre of gravity. Thus the regular triangle is always directly recognizable as the generating element of the hexagram.

The height **h** of the regular triangle stands to its side **a** in the relation $\mathbf{h}=\frac{\mathbf{a}\sqrt{3}}{2}$. Consequently, the ratio between the radius **r** of the circumscribed circle of the hexagon and the radius ρ of its inscribed circle is

$$\mathbf{r}:\rho=\mathbf{a}:\mathbf{h}=\mathbf{1}:\frac{\sqrt{3}}{2}.$$

It will be convenient to choose **D** to denote the diameter of the circum-
scribed circle of the hexagon — $\mathbf{D}=\mathbf{2r}$ XVIIa
— and Δ for the diameter of the inscribed circle —

$$\Delta=2\rho=\frac{2\mathbf{r}\sqrt{3}}{2}=\mathbf{r}\sqrt{3}.$$

In a *hexagram* the lineal dimensions of its generating *hexagon* are simply
XVIIc doubled. By this we mean to say that a larger hexagon drawn through the points of a hexagram parallel to the original hexagon has radii, heights, and so on, double the length of the analo-
XVIId gous original parts. Figure XVII,d gives a random selection of variations or amplifications that can be obtained by drawing concentric hexagrams and introducing circum- and inscribed hexagons to one and the same circle. It will be gathered that possibilities of modifying a plain basic hexagon are almost unlimited. But of course in an architectural design only a few of such possibilities will be actually made use of. So much may suffice for the hexagon and hexagram.

The most straightforward method
XVIIe of drawing a regular *octagon* is probably this: a circle is, by means of four diameters, divided into eight equal sectors, and the dividing points of the periphery are connected by eight chords. The area of the octagon then is made up of eight isosceles triangles, each formed by two radii $\mathbf{r}$ and a chord $\mathbf{s}$ (i.e. the octagon side), with an angle of 45° at the apex, and two angles of 67°30′ each at the base. The side $\mathbf{s}$, expressed through the radius, is $\mathbf{r}\sqrt{(2-\sqrt{2})}$, and the height ρ of the triangle (i.e. the inscribed radius of the octagon) is $\frac{\mathbf{r}}{2}\sqrt{(2+\sqrt{2})}$. But these somewhat unpleasant values are no concern of ours. As we shall encounter the *octagon* always, or nearly so, wedded to the *octagram*, we utilize the fact that the latter is easily derived from the regular tetragon or square, as follows:

We draw a square whose side be **A**, **XVIIf**
and over it the same figure again, turned by 45° about the centre of gravity. The eight sides **A** of these two squares intersect at eight points which form a regular octagon whose inscribed radius is $\frac{\mathbf{A}}{2}$. Externally attached to the eight octagon sides, which we call **a**, we find eight small isosceles rectangular triangles, the *teeth* of an *octagram*. Each of these triangles has **a**, the octagon side, as hypotenuse. Its two cathetes we will call **b**. Then $\mathbf{b}\sqrt{2} = \mathbf{a}$, or $\mathbf{b} = \frac{\mathbf{a}}{\sqrt{2}}$. Now, returning to the side **A** of the generating square, we see that it is composed of the octagon side **a** and twice the length **b**;

$$\mathbf{A} = \mathbf{b} + \mathbf{a} + \mathbf{b} = \mathbf{a} + 2\mathbf{b}.$$

Substituting $\frac{\mathbf{a}}{\sqrt{2}}$ for **b**, we have

$$\mathbf{A} = \mathbf{a}\left(\mathbf{1} + \frac{\mathbf{2}}{\sqrt{\mathbf{2}}}\right) = \mathbf{a}(\mathbf{1} + \sqrt{\mathbf{2}});$$

therefore $\boxed{\mathbf{a} = \frac{\mathbf{A}}{\mathbf{1} + \sqrt{\mathbf{2}}}}$. If, however, we want to express **A** through **b**, we have to write

$$\mathbf{A} = \mathbf{a} + 2\mathbf{b} = \mathbf{b}\sqrt{2} + 2\mathbf{b} = \mathbf{b}(2 + \sqrt{2});$$

therefore $\boxed{\mathbf{b} = \frac{\mathbf{A}}{\mathbf{2} + \sqrt{\mathbf{2}}}}$. We shall repeatedly meet with these comparatively simple relations between the side **A** of the generating square, the side **a** of the resulting octagon, and the length **b**, viz. the cathete of the tooth of the resulting octagram.

This plain octagram — we will call it octagram **A** — is now capable of similar amplifications to the hexagram, and the question arises what selections among these the ancient architects liked best.

A geometric pastime that offers itself readily is the consecutive inscribing of
XVIIg octagrams, i.e. the drawing of smaller parallel octagrams in such a way that each smaller figure touches the next larger one with the points of its teeth. Let the square side in the first of these smaller octagrams be **B**; it is easy to see that **B** equals the larger square side **A** divided by $\sqrt{2}$, since it is equal to half the diagonal of square **A**, viz.

$$\mathbf{B} = \frac{\mathbf{A}\sqrt{2}}{2} = \frac{\mathbf{A}}{\sqrt{2}}.$$

All the dimensions of the smaller octagram — we will call it octagram **B** — are, in relation to the original octagram **A**, proportionally reduced at the ratio $\frac{1}{\sqrt{2}} : 1$. Thus the *octagon*-side in octagram **B** equals $\mathbf{a} \times \frac{1}{\sqrt{2}}$; this however is $= \mathbf{b}$, the cathete in the tooth of the octagram **A**.

We can now proceed in the same way and draw a third octagram (dotted in figure XVII, g) inscribed to the second one, and call its square side **C**. Again the reduction at the ratio $\frac{1}{\sqrt{2}} : 1$ takes place, hence $\mathbf{C} = \frac{\mathbf{B}}{\sqrt{2}}$; since $\mathbf{B} = \frac{\mathbf{A}}{\sqrt{2}}$, **C** can be expressed through $\frac{\mathbf{A}}{\sqrt{2}\sqrt{2}}$; this however equals $\frac{\mathbf{A}}{2}$. In other words: by a twofold repetition of that inscribing (or 'telescoping') of octagrams we obtain a figure where all lineal dimensions are *halved* in relation to the initial octagram.

Nothing prevents us continuing this procedure *ad infinitum*, but the reader will, we believe, get tired of that somewhat uninspired game, and the ancient and mediaeval architects seem to have felt the same way. We shall see that another method of amplification or articulation, which we will now describe, was more familiar to them.

We start again from the octagram **A** (figure XVII, h, red). This time, however, we draw, in pairs of parallels, non-centric diagonals of the *octagon* perpendicular to the octagon sides; there are altogether four pairs or eight diagonals of this type. Their length is **A**. Each of these diagonals is cut four times by its fellows. The resulting internal figure is a miniature octagram (blue, hatched), concentric and parallel to octagram **A**. We see at once that the side of the generating square in this small figure equals **a**, the octagon side in octagram **A**, so we can call this inner or *core* figure octagram **a**. Since $\mathbf{a} = \frac{\mathbf{A}}{1 + \sqrt{2}}$, all lineal dimensions in octagram **a** are reduced from octagram **A** at the ratio $\mathbf{1} : (\mathbf{1} + \sqrt{\mathbf{2}})$.

We now 'telescope' into octagram **a** (blue) the next smaller figure (yellow, figure XVII, i) and find that the side of the generating square of this innermost octagram is $\frac{\mathbf{a}}{\sqrt{2}}$, consequently equal to **b**. Finally, we draw, as before in figure XVII, g, the telescoped octagram **B** (green) inside octagram **A** (red). The finished figure XVII,i thus shows four concentric octagrams in two separate groups, and it is evident that the *core*, formed by octagrams **a** and **b**, is the bodily reduction of the larger group — which we will call the 'frame' — formed of the octagrams **A** and **B**, in such a way that the two groups are inter-related at the ratio $(\mathbf{1} + \sqrt{\mathbf{2}}) : \mathbf{1}$.

The sides of the four generating squares are **A**, $\frac{\mathbf{A}}{\sqrt{2}}$, $\frac{\mathbf{A}}{\mathbf{1}+\sqrt{\mathbf{2}}}$ and $\frac{\mathbf{A}}{\mathbf{2}+\sqrt{\mathbf{2}}}$.[3]

If the reader has kept his patience until now, he will perhaps feel rewarded when he finds that this figure of two concentric double stars has a certain perfection, pleasing to the eye, and impressing itself on the memory.

Only a few more observations should be made at present: the distance between any two adjacent sides **A** (red) and **B** (green) equals that between **B** and **a** (blue), or a side **B** in any position is equidistant from the sides **A** and **a** parallel to it, that distance being $\frac{\mathbf{b}}{\mathbf{2}}$.

If we call the octagon side of octagram **a** α, and that of octagram **b** β, then the proportion obtains:

$$\mathbf{A} : \mathbf{a} : \alpha = \mathbf{B} : \mathbf{b} : \beta$$

A high importance attaches to what has been styled '$\frac{\pi}{4}$ lines.'[4] This means pairs of sides enclosing an angle of 45°. Such, for instance, are the centric diagonals of the octagon and octagram (see figure XVII, e). But many other pairs of diagonals or chords which enclose 45° or $\frac{\pi}{4}$ occur in our figures. Their inclination to the sides of the squares and octagons is either 22°30′ $\left(=\frac{\pi}{8}\right)$ or 67°30′ $\left(=\frac{3\pi}{8}\right)$. We now only point out the most momentous type of these $\frac{\pi}{4}$ lines, that with the apex in the point of one tooth of an octagram, and the basic points at the two ends of the opposite square side. We call that type '$\frac{\pi}{4}$ angle inscribed to octagram **A**'. **XVIIh**

In figure XVII,i such $\frac{\pi}{4}$ lines have been drawn (in black) for the three octagrams **A**, **B** and **a**.

Having reached this point, we feel we should apologize for introducing arithmetical expressions containing irrational values, viz. the roots of two and three. Should we, by any chance, be faced with the objection that to ancient and mediaeval architects all this would have been unintelligible, we possibly could not prove the contrary. It is true that the Greek mathematicians knew and used roots. But the Greeks and Romans did not know our numeral system, the Indian or 'Arabic' cyphers including 0 (zero) by which decimal fractions, rational and irrational, become possible, and so they could not express that e.g. $\mathbf{A}\sqrt{\mathbf{3}}$ equals **A** times 1·732 approximately. They would, therefore, resort to approximation by rational fractions, e.g. $\frac{\mathbf{17}}{\mathbf{12}} \sim \sqrt{\mathbf{2}}$ and $\frac{\mathbf{26}}{\mathbf{15}} \sim \sqrt{\mathbf{3}}$. The Arabic cyphers became first known in the north through a treatise of Al-Khowarizmi, written about 825 and translated into Latin by Adelard of Bath about 1120. Thus this new method of 'Algorism' would indeed have been within the reach of the architects of the Gothic period. But there is reason to doubt that the master-masons of the twelfth and following centuries could

[3]The numerical value of $\sqrt{\mathbf{2}}$ is 1·414, that of $\mathbf{1}+\sqrt{\mathbf{2}}$ is 2·414, that of $\mathbf{2}+\sqrt{\mathbf{2}}$ is 3·414 approximately.

[4]Alhard v. Drach, see Bibliography.

acquire sufficient knowledge of the new numeral system to compute their working dimensions. Not until the fifteenth century did the Arabic cyphers replace the Roman numerals in everyday practice; the general use of decimal fractions was established by S. Stevinus (1586) and J. Napier (1614).

But admitting this, we feel all the same quite sure that Greco-Roman, Early Christian, Byzantine, and Gothic architects were perfectly capable of drawing, articulating and varying hexagrams, octagrams, and other similar figures. It could be done, with every accuracy, on the drawing board or setting-out floor, in a purely geometric and draughtsmanlike way. The fact that these architects could not express irrational values through arithmetical symbols would only add to the mystic awe those geometrical forms inspired.

For our task, it will be an advantage to resort to arithmetical terms as much as possible, if only to cut short lengthy explanations in the course of geometrical analyses.

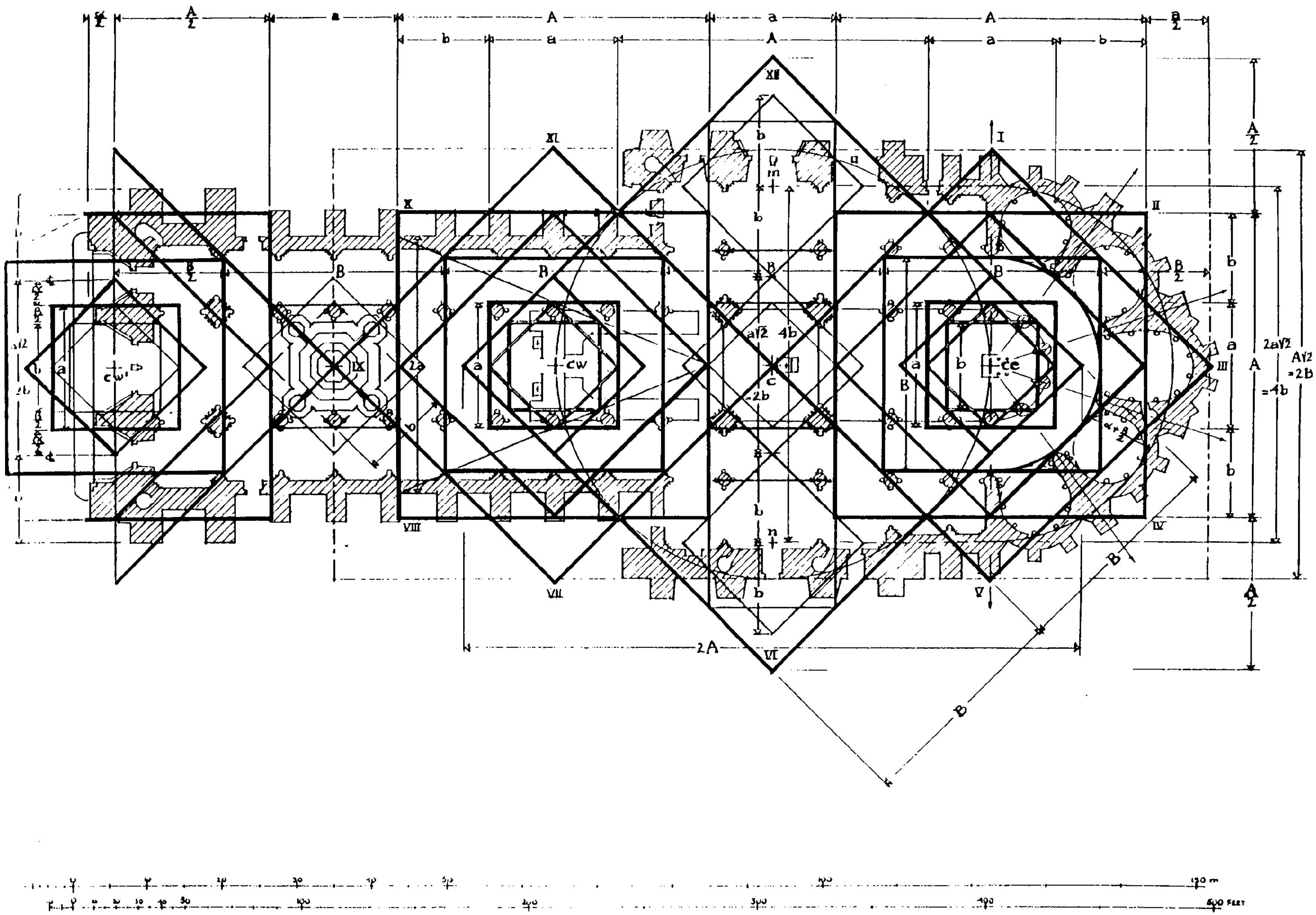
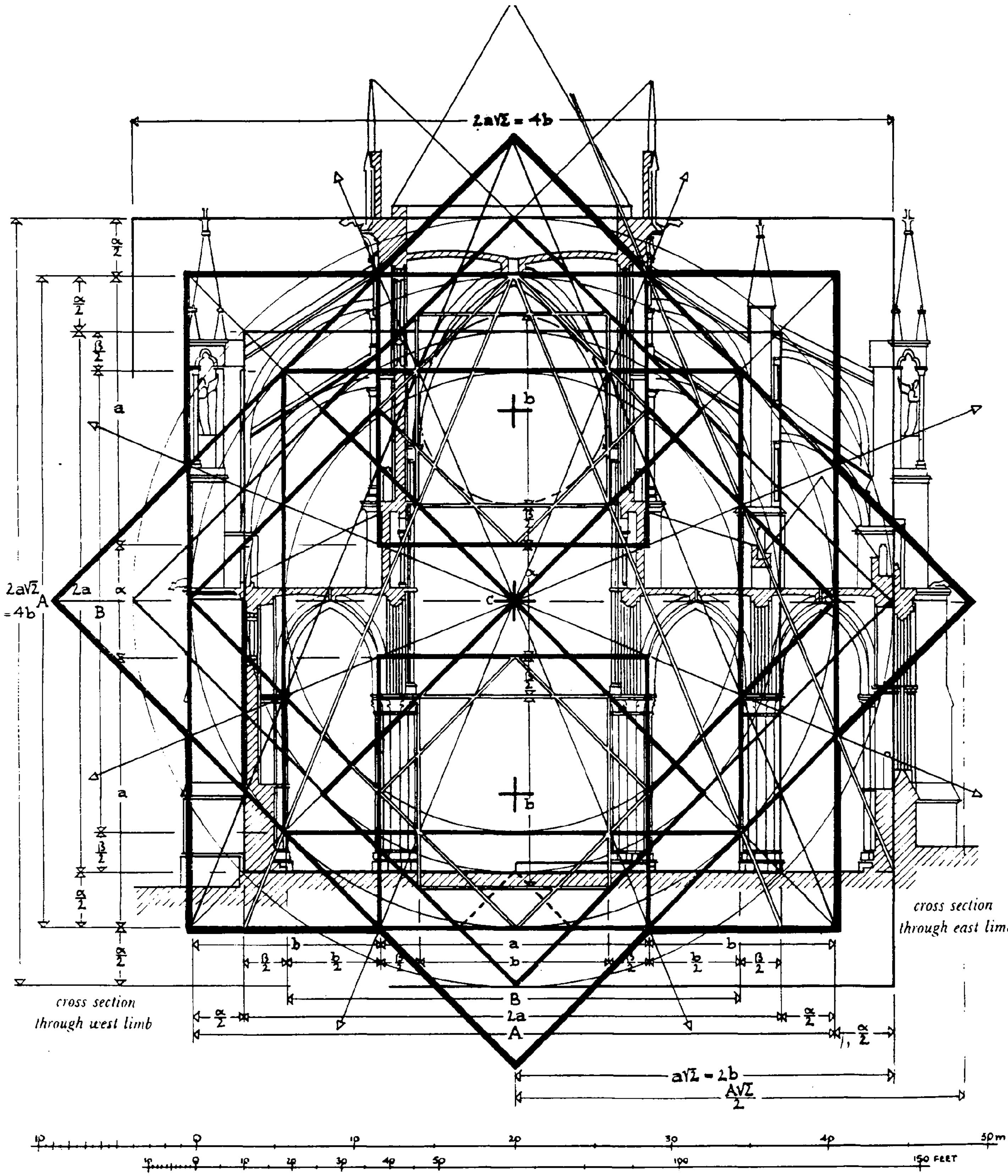
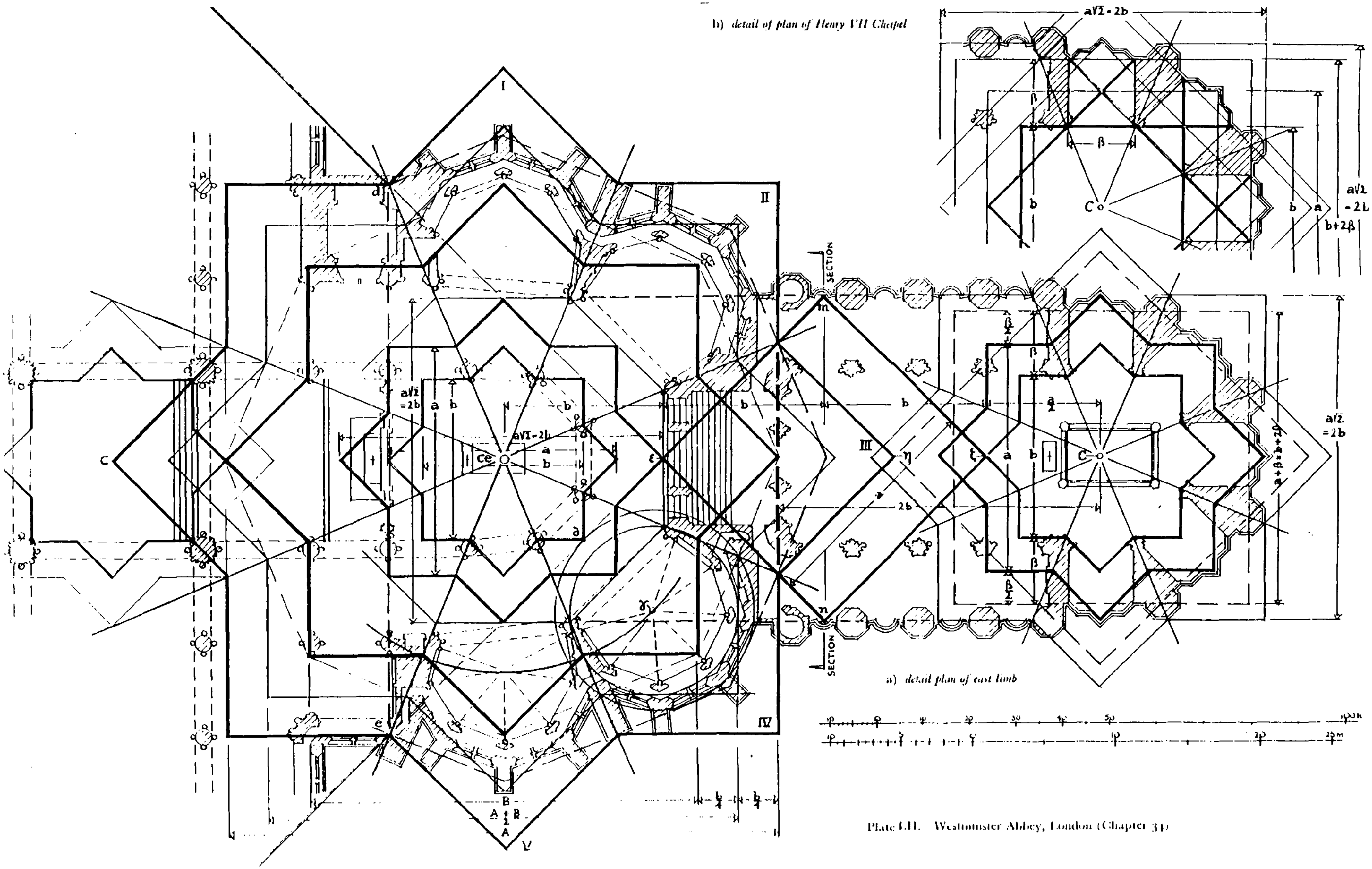


Plate XLVIII. Reims Cathedral, *plan* (Chapter 33).



2a√2 = 4b
cross section through west limb
cross section through east limb
a√2 = 2b
A√2/2
2a√2 = 4b
2a
A
B
a
b
B
2a
A
150 FEET
50 m



b) detail of plan of Henry VII Chapel

a) detail plan of east limb

Plate LII. Westminster Abbey, London (Chapter 34)